

A note on large deviations for the stable marriage of Poisson and Lebesgue with random appetites

Daniel Andrés Díaz Pachón*

August 23, 2010

Abstract

Let $\Xi \subset \mathbb{R}^d$ be a set of centers chosen according to a Poisson point process in \mathbb{R}^d . Let ψ be an allocation of \mathbb{R}^d to Ξ in the sense of the Gale-Shapley marriage problem, with the additional feature that every center $\xi \in \Xi$ has an appetite given by a nonnegative random variable α . Generalizing some previous results, we study large deviations for the distance of a typical point $x \in \mathbb{R}^d$ to its center $\psi(x) \in \Xi$, subject to some restrictions on the moments of α .

Key words: Poisson process, stable marriage, random appetite, large deviations.

1 Introduction

Let $\Xi \subset \mathbb{R}^d$ be a set of *centers* chosen according to a Poisson point process of law \mathbf{P} and expectation \mathbf{E} , whose intensity is $\lambda := \mathbf{E} [\Xi \cap [0, 1]^d]$. Let α be a nonnegative random variable (r.v.) with law \mathbb{P} and expectation \mathbb{E} called *appetite*. Every center $\xi \in \Xi$ will have a random appetite in such a way that appetites corresponding to different centers are independent of each other and identically distributed. The appetites are also independent of Ξ . With a little abuse of notation, but keeping in mind the independence, we will denote the appetite of a center ξ by α_ξ , when it does not lead to confusion (a formal construction is presented below). The elements $x \in \mathbb{R}^d$ will be called *sites*. Let ψ be a function of \mathbb{R}^d to $\Xi \cup \{\infty, \Delta\}$ called *allocation*. The goal of ψ is to allocate every site to a center. If ψ cannot warrant that allocation, then the site will be sent to the element ∞ . Also, there is a null set of sites under the Lebesgue measure \mathcal{L} which will be sent to Δ .

*Institute of Mathematical Statistics - Sao Paulo University. danielp@ime.usp.br

The *territory* of the center ξ will be given by $\psi^{-1}(\xi)$ and it satisfies the condition that $\mathcal{L}[\psi^{-1}(\xi)] \leq \alpha_\xi$. Whenever $\mathcal{L}[\psi^{-1}(\xi)] < \alpha_\xi$ the center will be *unsated*; if $\mathcal{L}[\psi^{-1}(\xi)] = \alpha_\xi$ it will be *sated*. Whenever $\psi(x) = \xi$ for some $\xi \notin \Delta$ we will say that x is *claimed*; if $\psi(x) = \infty$ we will call it *unclaimed*.

Consider a site x and a center ξ such that $\psi(x) \notin \{\xi, \Delta\}$. We will say that the site x *desires* the center ξ if $|x - \xi| < |x - \psi(x)|$ or if $\psi(x) = \infty$ (here and henceforth $|\cdot|$ is the Euclidean norm). If there exists $x' \in \psi^{-1}(\xi)$ such that $|x - \xi| < |x' - \xi|$, or if ξ is unsated, we will say that the center ξ *covets* the site x . A pair (x, ξ) will be *unstable* if x desires ξ and ξ covets x . If ψ does not produce any unstable pair it will be called *stable*. Finally, call \mathcal{C} the closure of the set of claimed sites.

Hoffman, Holroyd, and Peres gave an explicit construction of the function ψ for the case of constant appetite for every center in [4] and it was extended for the general case of random appetites in [1]. Formally, it is done using the stable marriage algorithm of Gale-Shapley (see [3]). Informally, all the centers will grow a ball at the same linear speed, claiming for each one every site not belonging to the territory of any other center. The process continues for each center ξ up to the point in which ξ is sated (has claimed for itself a territory with volume α_ξ). If its appetite is never satisfied the ball will continue growing to ∞ trying to find unclaimed sites and the center will be unsated.

We proceed now to describe the probability space based on a construction proposed by Mester and Roy for the Poisson Boolean model (see [7], pp. 16–17): Let Ξ be defined on a probability space $(\Omega_1, \mathcal{F}_1, \mathbf{P})$. Let $(\Omega_2, \mathcal{F}_2, P)$ be the probability space on which the *iid* r.v.'s $\{\alpha(n, z), n \in \mathbb{N}, z \in \mathbb{Z}^d\}$, distributed as α , are defined (thus, Ω_2 has a product structure such that each marginal has law \mathbb{P}). Set $\Omega = \Omega_1 \times \Omega_2$, equip it with the usual product σ -algebra and product measure $\mathcal{P} = \mathbf{P} \times P$. The configuration of appetites of centers corresponding to (ω_1, ω_2) is obtained using *binary cubes of order* n :

$$K(n, z) := \prod_{i=1}^d (z_i 2^{-n}, (z_i + 1) 2^{-n}] \text{ for all } n \in \mathbb{N} \text{ and } z \in \mathbb{Z}^d.$$

Now, each center $\xi \in \Xi$ is contained in a unique binary cube of order n , $K(n, z(n, \xi))$, and \mathbf{P} -a.s., for each $\xi \in \Xi$ there is a unique smallest number $n_0 = n_0(\xi)$ such that $K(n_0, z(n_0, \xi))$ contains no other centers of Ξ . The appetite of the ball centered at ξ is given by $\alpha(n_0, z(n_0, \xi))$, which will be noted for simplicity as α_ξ when it does not lead to confusion, as it was said at the beginning.

Thus, the product structure of Ω warrants the independence between appetites and centers and the product structure of Ω_2 warrants the independence among the appetites for different centers.

Remark. As it is noted in [7], maybe the first construction that comes to mind is one in which the centers are ordered linearly following some determined law, and to construct in one probability space the point process Ξ and *iid* r.v.'s $\alpha_1, \alpha_2, \dots$, and construct a realization by assigning α_i to the i -th center of Ξ . The problem with this setting is that we will need to use some ergodic results but the ordering of the centers is not preserved with translations.

The model can be denoted as (Ξ, α, ψ) with joint law \mathcal{P} and expectation \mathcal{E} . Since ψ is *translation-equivariant* (i.e., $\psi_\Xi(x) = \xi$ implies that $\psi_{T\Xi}(Tx) = T\xi$, for every translation $T \in \mathbb{R}^d$ and where ψ_Ξ is an allocation to the support $[\Xi]$ of Ξ), we have that \mathcal{P} is translation invariant. Its *Palm version* $(\Xi^*, \alpha^*, \psi^*)$, with law \mathcal{P}^* and expectation operator \mathcal{E}^* , is conditioned to have a center on the origin. Note that since α is independent of Ξ , then α^* and Ξ^* are also independent and $\alpha = \alpha^*$.

By a result of Thorisson in [9] (which was the beginning of the current research in allocations), Ξ and Ξ^* can be coupled so that almost surely one is a translation of the other. Therefore, ψ_{Ξ^*} is defined \mathcal{P}^* -a.s. Finally, remember that since Ξ is a Poisson process, Ξ^* is a Poisson process with an added point at the origin. (See a similar reasoning in [4] p. 1256 and [6] pp. 8–9).

Defined this way, there are at least two quantities natural to consider in the model: $X' := |x - \psi(x)|$, which by translation-invariance happens to have the same law as $X = |\psi(0)|$; and $R_{\psi_\Xi}(\xi) = \text{ess sup}_{x \in \psi^{-1}(\xi)} |\xi - x|$: the radius of $\psi^{-1}(\xi)$. Thus, the radius of a typical center can be defined as $R^* = R_{\psi_{\Xi^*}}(0)$.

Continuing with what was done in [2] with percolation, we generalize here some results of the stable marriage of Poisson and Lebesgue when the appetite is random. The results presented in this article were first obtained in [5] for the case of constant appetite. We use the same methods to obtain large deviations for the case of random appetites, but of course there is a weakening: generally, when the appetites are constant the decay on the laws of R^* and X is exponential in $(R^*)^d$ and X^d , respectively; but now, with random appetites, the decay is polynomial. We will keep the notation as consonant with [5] as possible.

Besides the existence of the stable allocation for a discrete set of centers, in [1] there were some generalizations of previous results in [4] and [5]. Since most of proofs of these results remain almost unchanged (but in the random appetite case the results will hold \mathcal{P} -a.s.), here we mention some of them without proof and refer the reader directly to [4].

Theorem 1.1 (Almost sure uniqueness). Let Ξ be a Poisson point process with law \mathbf{P} and finite intensity λ , where each center ξ chooses its appetite with law \mathbb{P} , then there exists a unique stable allocation for \mathcal{L} -a.e. $x \in \mathbb{R}^d$, \mathcal{P} -a.s.

Theorem 1.2 (Phase transitions). Let Ξ be a Poisson point process with law \mathbf{P} and intensity $\lambda \in (0, \infty)$.

- If $\lambda \mathbb{E}\alpha < 1$ (**subcritical**), then all the centers are sated but there exists an infinite volume of unclaimed sites, \mathcal{P} -a.s.
- If $\lambda \mathbb{E}\alpha = 1$ (**critical**), then all the centers are sated and \mathcal{L} -a.e. site is claimed, \mathcal{P} -a.s.
- If $\lambda \mathbb{E}\alpha > 1$ (**supercritical**), then there are unsated centers but \mathcal{L} -a.e. site is claimed, \mathcal{P} -a.s.

A set A is an *essential subset* of B if $\mathcal{L}(B \setminus A) = 0$. An allocation will be called *canonical* if for every site x such that there exists $\xi \in \Xi \cup \infty$ satisfying that x has a neighborhood which is an essential subset of $\psi^{-1}(\xi)$, then $\psi(x) = \xi$; if there is no such ξ , then $\psi(x) = \Delta$.

Theorem 1.3 (Canonical allocations). It is possible to construct a stable allocation to Ξ which is also canonical and minimizes the \mathcal{L} -null set Δ .

Sets of centers satisfying that they have a unique stable allocation \mathcal{L} -a.e. and a unique canonical allocation are called *benign* sets. In particular, Ξ is benign by Theorems 1.1 and 1.3.

Now, for sets of centers Ξ_1, Ξ_2, \dots and Ξ , write $\Xi_n \Rightarrow \Xi$ if for any compact $K \subset \mathbb{R}^d$ there exists N such that for $n > N$ we have $\Xi_n \cap K = \Xi \cap K$. For allocations ψ_1, ψ_2, \dots and ψ write $\psi_n \rightarrow \psi$ a.e., if for \mathcal{L} -a.e. $x \in \mathbb{R}^d$ we have $\psi_n(x) \rightarrow \psi(x)$ in the one-point compactification $\mathbb{R}^d \cup \{\infty\}$.

Exactly as it was done with $\{\alpha_\xi\}_{\xi \in \Xi}$, for $i \in \mathbb{N}$ we are going to construct a family of *iid* random variables $\{\alpha_i(\xi)\}_{\xi \in \mathbb{N}}$ independent of α , independent of every set of centers, and with the same distribution as α (Formally, we repeat the process above to obtain r.v.'s $\alpha_i(n_0, z(n_0, \xi))$, for $n_0 \in \mathbb{N}$ and

$z \in \mathbb{Z}^d$). Then, to the center $\xi \in \Xi_i$ corresponds the appetite $\alpha_i(\xi)$. We can define the random variables $\hat{\alpha}_i(\xi)$ as follows:

$$\hat{\alpha}_i(\xi) = \begin{cases} \alpha_i(\xi) & \text{for all } \xi \notin \Xi \\ \alpha_\xi & \text{otherwise.} \end{cases}$$

Thus, as required, $\{\hat{\alpha}_i(\xi)\}_{\xi \in \mathbb{N}}$ is an *iid* set of r.v.'s distributed as α . We are going to use the following natural extension of Theorem 5 in [5]:

Theorem 1.4 (Continuity). Let Ξ_1, Ξ_2, \dots and Ξ be benign sets of centers with appetites given by the $\hat{\alpha}_i(\xi)$ for $\xi \in \Xi_i$ in Ξ_i and α for all the centers in Ξ . Write $\psi_n = \psi_{\Xi_n}$ $\psi = \psi_\Xi$ for their canonical allocations. If $\Xi_n \Rightarrow \Xi$, then $\psi_n \rightarrow \psi$ a.e., \mathbb{P} -a.s.

Remark. Theorems 1.1, 1.2, and 1.3 have more general statements in [4] (and [1]). Theorem 1.1 was proved for every translation-invariant point process, and Theorems 1.2 and 1.3 were proved for every point process ergodic under translations. For our interest, the versions so stated will suffice. The proof of Theorem 1.4 is essentially the same as the proof of Theorem 5 in [5] once a classical coupling is constructed via the r.v.'s $\hat{\alpha}_i(\xi)$.

There are some additional things needed for the proofs but they will be introduced in situ. For now, what we have suffices to state the results in this article. Finally, with a little abuse of notation to make the article more readable, we will use \mathcal{P} and \mathcal{E} , except when it will be explicitly clear that we are dealing with Ξ and α .

2 Statement of the results

Our first result determines explicit bounds of the appetites under certain restrictions of α . The next two theorems establish results for the upper bounds of X and R^* in the uncritical phases.

Theorem 2.1. Let Ξ be a Poisson process in \mathbb{R}^d with intensity 1 and consider a stable allocation with appetite α .

- (i) Let α be a r.v. with mean $\mu > 2^d$, $d > 1$ and variance σ^2 . Then, given $\delta \in (0, d)$, we have

$$\mathcal{E}X^{d-\delta} < \infty.$$

(ii) Let α be a r.v. bounded from above by M . Assume that α has mean $\mu < 2^{-d}$ and variance σ^2 . Then, for every $\varepsilon \in (0, d/2)$, we have that $\mathcal{E}^* e^{c(R^*)^\varepsilon} < \infty$, for some $c > 0$.

Theorem 2.2 (Supercritical upper bound). Let Ξ be a set of centers of intensity $\lambda > 1$ and let α be the appetite with mean 1 and variance σ^2 . Then, for $\beta \in (0, d)$, we have that $\mathcal{E}X^{d-\beta} < \infty$.

Theorem 2.3 (Subcritical upper bound). Let Ξ be a Poisson process of intensity $\lambda < 1$ in \mathbb{R}^d with $d > 2$. Let α be the appetite with mean 1 and finite variance σ^2 . Then, for $\beta \in (0, d/2)$, we have that $\mathcal{E}^*(R^*)^{d/2-\beta} < \infty$.

3 Explicit bounds

We start from mentioning two results of Nagaev for the large deviation of the sums of independent r.v. and two known results of Chernoff on the large deviations of a Poisson r.v.

Proposition 3.1 (Corollary 1.8 in [8]). Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of *iid* r.v. with $\mathcal{E}X_1 = 0$ and variance σ^2 . Define $A_2^+ := \int_{u \geq 0} u^2 dF(u)$ and take $x > 0$, then

$$\mathcal{P} \left[\sum_{i=1}^n X_i \geq x \right] \leq \frac{4nA_2^+}{x^2} + \exp \left\{ \frac{-8x^2}{ne^2\sigma^2} \right\}$$

Proposition 3.2 (Corollary 1.12 in [8]). Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of *iid* r.v. with $\mathcal{E}X_1 = 0$, $X_1 \leq M$, and variance σ^2 . Let $S_n = \sum_{i=1}^n X_i$ and take $x > 0$, then

$$\mathcal{P}[S_n \geq x] \leq \exp \left[x/M - (x/M + n\sigma^2/M^2) \log \left(\frac{xM}{n\sigma^2} + 1 \right) \right].$$

Lemma 3.1 (Chernoff inequalities for Poisson r.v.). Let N be a Poisson r.v. with mean $\lambda \in (0, +\infty)$, then

$$\mathcal{P}[N \geq a] \leq e^{-\lambda g(\lambda/a)} \quad \text{for } a > \lambda \quad (1)$$

$$\mathcal{P}[N \leq a'] \leq e^{-\lambda g(\lambda/a')} \quad \text{for } a' < \lambda \quad (2)$$

where $g(x) = (x - 1 - \log x)/x$ (so $g(x) \rightarrow \infty$ when $x \rightarrow 0$).

Proof of Theorem 2.1. To prove (i) let Z be the amount of centers of Ξ in the ball $B(0, r)$, $r > 0$. Thus, Z is Poisson with mean $\pi_d r^d$, where π_d is the volume of the unit ball in d dimensions. We prove first that

$$\sum_{\xi \in B(0, r)} \alpha_\xi \geq \pi_d 2^d r^d \Rightarrow X < r. \quad (3)$$

Suppose on the contrary that $\psi(0) = X \geq r$. Then 0 desires every center inside $B(0, r)$. Taking now the antecedent in (3) we have two options: Either every center in $B(0, r)$ is unsated, or there is at least one center $\xi \in B(0, r)$ with \mathcal{L} -non-null territory T outside $B(0, 2r)$; i.e., $T \subset \psi^{-1}(\xi) \cap [\mathbb{R}^d \setminus B(0, 2r)]$ and $\mathcal{L}T > 0$ for some $\xi \in B(0, r)$. In the first case, since every center $\xi' \in B(0, r)$ is unsated, then ξ' covets 0 by definition, thus $(0, \xi')$ is an unstable pair. In the second case, for every $x \in T$ we have that $|\xi - x| > r > |\xi| = |\xi - 0|$. Thus, ξ covets 0 and $(0, \xi)$ is an unstable pair. A contradiction, and (3) is established. Then,

$$\begin{aligned} \mathcal{P}[X > r] &\leq \mathcal{P} \left[\sum_{\xi \in B(0, r)} \alpha_\xi < \pi_d 2^d r^d \right] \\ &\leq \mathcal{P} \left[\sum_{i=1}^Z \alpha_i \leq \pi_d 2^d r^d \mid Z \geq K \right] \mathcal{P}[Z \geq K] + \mathcal{P}[Z < K] \\ &\leq \mathcal{P} \left[\sum_{i=1}^{\lfloor K \rfloor} \alpha_i \leq \pi_d 2^d r^d \right] + \mathcal{P}[Z < K] \\ &= \mathcal{P} \left[\sum_{i=1}^{\lfloor K \rfloor} -\alpha'_i \geq -a \right] + \mathcal{P}[Z < K] \\ &\leq \frac{4\lfloor K \rfloor A_2^+}{(-a)^2} + \exp \left\{ \frac{-8(-a)^2}{\lfloor K \rfloor e^2 \sigma^2} \right\} + e^{-\pi_d r^d g(\pi_d r^d / \lfloor K \rfloor)}, \end{aligned} \quad (4)$$

where $\alpha'_k = \alpha_k - \mu$ for $k = 0, 1, \dots, K$, $K = \pi_d(r^d - r^{d/2+\varepsilon})$ for small positive ε ; $a = \pi_d 2^d r^d - K\mu = \pi_d r^d(2^d - \mu - \mu r^{\varepsilon-d/2})$. Then, by the definition of μ , $a < 0$ when $r^{d/2-\varepsilon} > \mu/(\mu - 2^d)$. The first two terms in (4) result from Proposition 3.1 and the third term results from the Chernoff inequality (2) in Lemma 3.1. Given that $g(x) \sim (x-1)^2/2$ when $x \rightarrow 1$, we have that $g(r^d/(r^d - r^{d/2+\varepsilon})) \sim r^{-d+2\varepsilon}/2$ when $r \rightarrow \infty$. Thus, for r large enough,

$$\begin{aligned}\mathcal{P}[X > r] &\leq c_1 r^{-d} + e^{-c_2 r^d} + e^{-c_3 r^{2\varepsilon}} \\ &\leq 3c_1 r^{-d}.\end{aligned}$$

where $c_3 < \pi_d/2$, $c_2 < 8/(e^2 \sigma^2)$, and $c_1 < 4A_2^+$. To finish the proof note that, given $\delta \in (0, d)$,

$$\mathcal{E} X^{d-\delta} = \int_0^\infty \mathcal{P}\left[X \geq t^{1/(d-\delta)}\right] dt \leq 1 + 3c_1 \int_1^\infty t^{-d/(d-\delta)} dt < \infty. \quad (5)$$

In order to prove (ii) let Z' be the amount of centers of Ξ^* in $B(0, 2r)$. Then $Z' - 1$ is distributed as a Poisson with intensity $\pi_d 2^d r^d$. If $Z' < M^{-1} \pi_d r^d$ we have that $\sum_{\xi \in B(0, 2r)} \alpha_\xi < \pi_d r^d$. Therefore, there is a \mathcal{L} -non-null set L in $B(0, r)$ such that its elements do not belong to the territory of the centers of Ξ^* in $B(0, 2r) \cap \Xi^*$. Hence, the stability of the pair $(x, 0)$ implies that the appetite of 0 was already satisfied inside the closed ball $B[0, |x|]$. Then

$$R^* < |x| < r.$$

We proceed as in part (i), to obtain that

$$\begin{aligned}\mathcal{P}[R^* > r] &\leq \mathcal{P}\left[\sum_{k=0}^{Z'} \alpha_k > \pi_d r^d\right] \\ &\leq \mathcal{P}\left[\sum_{k=0}^{Z'} \alpha_k > \pi_d r^d \mid Z' \leq K\right] \mathcal{P}[Z' \leq K] + \mathcal{P}[Z' > K] \\ &\leq \mathcal{P}\left[\sum_{k=0}^{\lceil K \rceil} \alpha'_k > a\right] + \mathcal{P}[Z' > K]\end{aligned} \quad (6)$$

where $\alpha'_k = \alpha_k - \mu$ for $k = 0, 1, \dots, K$; $K = \pi_d 2^d (r^d + r^{d/2+\varepsilon})$; $a = \pi_d r^d - K\mu = \pi_d r^d (1 - 2^d \mu (1 + r^{-d/2+\varepsilon}))$; and $\varepsilon \in (0, d/2)$. Then $a > 0$ when $r^{d/2-\varepsilon} > 2^d \mu / (1 - 2^d \mu)$. The appetite of the center at the origin is noted by α_0 . Now, after Proposition 3.2, the first term in (6) can be bounded by

$$\exp\left\{a/M - [a/M + (\lceil K \rceil + 1) \sigma^2 / M^2] \log\left(\frac{aM}{(\lceil K \rceil + 1) \sigma^2} + 1\right)\right\}, \quad (7)$$

where we will assume without loss of generality that $M > (e-1)\sigma^2(\lceil K \rceil + 1)/a$, in order to make the expression inside the logarithm greater than 1. Also, using Lemma 3.1, the second term in (6) can be bounded by

$$\exp\{-\pi_d 2^d r^d g(\pi_d 2^d r^d / K)\}. \quad (8)$$

Note that $g(r^d/(r^d + r^{d/2+\varepsilon})) \sim r^{-d+2\varepsilon}/2$. With those considerations in mind, using (6), (7), and (8), we obtain

$$\begin{aligned} \mathcal{P}[R^* > r] &\leq e^{a/M - [a/M + K\sigma^2/M^2]} + e^{-c_1 r^\varepsilon} \\ &\leq e^{-c_2 r^d} + e^{-c_1 r^\varepsilon} \\ &\leq 2e^{-c_3 r^\varepsilon}. \end{aligned}$$

where $c_1 = c_1(d) < \pi_d 2^{d-1}$, $c_2 = c_2(d) < \pi_d 2^d \sigma^2 / M^2$, and $c_3 = \min\{c_2, c_1\}$. \square

4 Supercritical bound

Given a benign set $\Xi \subset \mathbb{R}^d$ and a measurable set $A \subset \mathbb{R}^d$, let Ξ'_A be a random set of centers which is the union of $\Xi \cap A$ and a Poisson process with intensity λ in $\mathbb{R}^d \setminus A$. Call $F_{\lambda,A}$ the law of Ξ'_A and let $Q(L) := [-L, L]^d$. With these concepts at hand we will introduce the following definition and results which will equip us to prove Theorem 2.2.

Definition 4.1 (Decisive set). Given a benign set Ξ , the measurable set $A \subset \mathbb{R}^d$ is Ξ -**decisive** for a site x if for every $\lambda \in (0, \infty)$ we have that $F_{\lambda,A}$ -a.s., Ξ'_A is a benign set and $\psi_{\Xi'_A}(x) = \psi_\Xi(x)$.

Proposition 4.1. Let $\hat{\alpha}_n(\xi)$ and α , Ξ_n and Ξ , and ψ_n and ψ , be the appetites, sets of centers and allocations, respectively, as in Theorem 1.4. If $\psi(z) = \xi \in \Xi$ and z is not equidistant from any two centers of Ξ , then $\psi_n(z) \rightarrow \xi$.

Proof. It corresponds to Proposition 19 in [5]. It is a refinement of Theorem 1.4. \square

Lemma 4.1. Let $\mathbb{E}\alpha = 1$ and let Ξ be a Poisson process of intensity $\lambda \geq 1$. Then, \mathbf{P} -a.s. and \mathbb{P} -a.s., there exists $L < \infty$ such that $Q(L)$ is Ξ -decisive for 0.

Proof. Since $\lambda \geq 1$ we have by Theorem 1.2 that 0 is claimed a.s. Also, 0 is not equidistant from any two centers, a.s. On the complement of the event in the lemma, for each L there exists a benign Ξ_L that agrees with Ξ on $Q(L)$ such that $\psi_{\Xi_L}(0) \neq \psi_{\Xi}(0)$. But by Proposition 4.1 this is an event with null probability \mathbb{P} , since when this happens we have $\psi_{\Xi_L}(0) \rightarrow \psi_{\Xi}(0)$ as $L \rightarrow \infty$, \mathbb{P} - a.s. \square

Lemma 4.2 (Decisive cubes). Let $\mathbb{E}\alpha = 1$ and let Ξ be a Poisson process of intensity $\lambda \geq 1$. For every $\varepsilon > 0$ there exists $M < \infty$ such that

$$\mathcal{EL}[x \in Q(M) : Q(M) \text{ is not } [\Xi]\text{-decisive for } x] < \varepsilon(2M)^d.$$

Proof. Fix $\varepsilon > 0$. Let U^L be the random set of sites x for which $Q(L) + x$ is not Ξ -decisive. Then U^L is translation-invariant in law, and by Lemma 4.1, we may fix L large enough so that it has intensity less than $\varepsilon/2$. For M sufficiently large we have

$$\mathbb{E}\mathcal{L}\{U^L \cap Q(M)\} < (\varepsilon/2)(2M)^d,$$

thus

$$\begin{aligned} \mathbb{E}\mathcal{L}\{x \in Q(M) : Q(M) \text{ is not } \Xi\text{-decisive for } x\} \\ < (\varepsilon/2)(2M)^d + (2M)^d - (2M - 2L)^d \\ < \varepsilon(2M)^d. \end{aligned}$$

\square

Lemmas 4.1, 4.2 and Proposition 4.1 were introduced in [5]. We transcribed them here for the sake of thoroughness and contextualization. Now we have what we need to proceed with the proof of Theorem 2.2:

Proof of Theorem 2.2. Note first that if

$$\text{there exists } \xi \in [\Xi] \cap B(0, r) \text{ with } \mathcal{L}[\psi^{-1}(\xi) \cap B(0, 2r)] < \alpha_\xi, \quad (9)$$

then $X \leq r$. Otherwise, if $X > r$, then ξ covets 0 and 0 desires ξ , and thus $(0, \xi)$ is unstable. Hence,

$$\mathcal{P}[X > r] \leq \mathcal{P}\{\mathcal{L}[\psi^{-1}(\xi) \cap B(0, 2r)] = \alpha_\xi, \text{ for all } \xi \in \Xi \cap B(0, r)\}.$$

Thus, it is enough to prove that the event at the right hand of the former inequality has polynomial decay. For a given λ let

$$\varepsilon = \frac{\lambda - 1}{10 \cdot 2^d} \wedge 1,$$

and let $M = M(\lambda, \varepsilon)$, as in Lemma 4.2.

For all $r > 0$ we are going to place disjoint cubes congruent to $Q(M)$ in $B(0, 2r) \setminus B(0, r)$. Define $Q_z = Q(M) + 2Mz$, for $z \in \mathbb{Z}^d$ and define

$$Y_z = \mathcal{L}(x \in Q_z : Q_z \text{ is not } \Xi\text{-decisive for } x).$$

Define also

$$I = I(r) = \left\{ z \in \mathbb{Z}^d : Q_z \subset B(0, 2r) \setminus B(0, r) \right\},$$

this is the index set of the cubes which are completely inside the shell. Finally, let

$$S = S(r) = [B(0, 2r) \setminus B(0, r)] \setminus \bigcup_{z \in I} Q_z$$

be the part of the shell which is uncovered by $\bigcup_{z \in I} Q_z$. For large enough r we have that

$$\mathcal{L}S < \varepsilon \pi_d r^d. \quad (10)$$

To prove this, it suffices to prove that

$$\mathcal{L} \left[B(0, r + \sqrt{2}M) \setminus B(0, r) \right] + \mathcal{L} \left[B(0, 2r) \setminus B(0, 2r - \sqrt{2}M) \right] < \varepsilon \pi_d r^d,$$

for large enough r . Which is equivalent to prove that $\pi_d[(r + \sqrt{2}M)^d - r^d] + \pi_d[2^d r^d - (2r - \sqrt{2}M)^d] < \varepsilon \pi_d r^d$ for large enough r . But this is clear since every factor of π_d at the left side of the inequality has order r^{d-1} .

Consider now the next events:

$$E = \left\{ \sum_{\xi \in B(0, r)} \alpha_\xi > (\lambda - \varepsilon) \pi_d r^d \right\};$$

$$G = \left\{ \sum_{z \in I} Y_z < 4\varepsilon 2^d \pi_d r^d \right\}$$

and suppose that E and G occur. Then

$$\begin{aligned} \mathcal{L}[x \in B(0, 2r) : \psi(x) \in B(0, r)] &\leq \sum_{z \in I} Y_z + \pi_d r^d + \mathcal{L}S \\ &\leq (4\varepsilon 2^d + 1 + \varepsilon) \pi_d r^d \\ &< (\lambda - \varepsilon) \pi_d r^d \end{aligned}$$

$$< \sum_{\xi \in B(0,r)} \alpha_\xi,$$

where the third inequality is obtained because of the definition of ε . Then the event in (9) occurs when E and G occur. So it is enough to study the tails of $\mathcal{P}[E^C]$ and $\mathcal{P}[G^C]$ to conclude that the decay of X will be at least the one of these two with heavier tail.

We start with $\mathcal{P}[G^C]$. Note that $(Y_z)_{z \in I}$ is a set of *iid* r.v. Furthermore, $Y_z \in (0, 2^d M^d)$. Thus, Y_z has finite mean μ_Y (in fact, $\mu_Y < \varepsilon(2M)^d/2$) and variance σ_Y^2 . Let $\#I$ be the cardinality of the set I (note that $\#I$ has order r^d) and let $K = \pi_d r^d (2^d - 1)/(2M)^d$, which would be the maximum number of cubes with side $2M$ in the shell $B(0, 2r) \setminus B(0, r)$. Then, using the Nagaev inequality of Proposition 3.1, we obtain that, for large r ,

$$\begin{aligned} \mathcal{P} \left[\sum_{z \in I} Y'_z \geq y \right] &\leq \mathcal{P} \left[\sum_{z=1}^{\lceil K \rceil} Y'_z \geq y \right] \\ &\leq 4 \lceil K \rceil A_2^+ / y^2 + \exp\{-8y^2 / (\lceil K \rceil e^2 \sigma_Y^2)\} \\ &\leq c' r^{-d} \end{aligned}$$

where $Y'_z = Y_z - \mu_Y$, $y = 4\varepsilon 2^d \pi_d r^d - (\#I)\mu_Y$.

As for $\mathcal{P}[E^C]$, let $\alpha'_k = \alpha_k - 1$ for $k = 1, \dots, K$, $K = \pi_d(\lambda r^d - r^{d/2+\delta})$, where δ is a small enough positive real. Let $a = (\lambda - \varepsilon)\pi_d r^d - K = \pi_d r^{d/2+\delta}(1 - \varepsilon r^{d/2-\delta})$, which will be negative for $r > \varepsilon^{-2/(d-2\delta)}$. Finally, let N be a Poisson r.v. with intensity $\lambda \pi_d r^d$. Then we have,

$$\begin{aligned} \mathcal{P}[E^C] &= \mathcal{P} \left[\sum_{\xi \in B(0,r)} \alpha_\xi \leq (\lambda - \varepsilon)\pi_d r^d \right] \\ &\leq \mathcal{P} \left[\sum_{i=1}^N \alpha_i \leq (\lambda - \varepsilon)\pi_d r^d \mid N \geq K \right] \mathcal{P}[N \geq K] + \mathcal{P}[N < K] \\ &\leq \mathcal{P} \left[\sum_{i=1}^{\lfloor K \rfloor} \alpha_i \leq (\lambda - \varepsilon)\pi_d r^d \right] \mathcal{P}[N \geq K] + \mathcal{P}[N < K] \\ &\leq \mathcal{P} \left[\sum_{i=1}^{\lfloor K \rfloor} -\alpha'_i \geq -a \right] + \mathcal{P}[N < K]. \end{aligned}$$

By the Chernoff inequality (2) in Lemma 3.1 we obtain for the second term in the last inequality that

$$\begin{aligned}\mathcal{P}[N < K] &\leq e^{-\lambda\pi_d r^d g(\lambda\pi_d r^d/K)} \\ &\leq e^{-c_1 r^{2\delta}}\end{aligned}\tag{11}$$

for some $c_1 > 0$. And for the first term, using Proposition 3.1, we obtain that

$$\begin{aligned}\mathcal{P}\left[\sum_{i=1}^{\lfloor K \rfloor} -\alpha'_i \geq -a\right] &\leq \frac{4\lfloor K \rfloor A_2^+}{a^2} + \exp\left\{\frac{-8a^2}{\lfloor K \rfloor e^2 \sigma^2}\right\} \\ &\leq c_3 r^{-d} + e^{-c_2 r^d}.\end{aligned}\tag{12}$$

Thus, from (11) and (12) we obtain

$$\begin{aligned}\mathcal{P}[E^C] &\leq c_3 r^{-d} + e^{-c_2 r^d} + e^{-c_1 r^d} \\ &\leq c_4 r^{-d}.\end{aligned}$$

Since both terms have polynomial decay, we obtain the result after an application of (5). \square

5 Subcritical bound

In order to prove Theorem 2.3, we will need Proposition 5.1 below. Although the proof of this Proposition is almost identical to the one stated by Hoffman, Holroyd and Peres (see Lemmas 14 and 18, and Corollary 15 in [5]), we reproduce it here for the sake of clarity and completeness, with the suitable minor changes needed in this case.

Lemma 5.1. Suppose $\Xi_n \Rightarrow \Xi$ and let $\alpha'_n(\xi)$ be the appetite of the centers $\xi \in \Xi_n$ and α in Ξ as in Theorem 1.4. Then $\psi_n \rightarrow \psi$, almost everywhere. If there is a set A of positive volume such that every $z \in A$ desires ξ under ψ , then for n sufficiently large, ξ is sated in ψ_n and

$$\limsup_{n \rightarrow \infty} R_{\psi_n}(\xi) \leq \text{ess inf}_{z \in A} |z - \xi| < \infty.\tag{13}$$

Proof. It is identical to the proof of Lemma 18 in [5]. \square

Definition 5.1 (Replete set). Given a benign set Ξ , we say that a measurable set $A \subset \mathbb{R}^d$ is Ξ -**replete** for a center ξ if for every $\lambda \in (0, \infty)$ we have that, $F_{\lambda, A}$ -a.s., Ξ'_A is benign and $\mathcal{L}\left(\psi_{\Xi'_A}^{-1}(\xi) \cap A\right) = \alpha_\xi$.

Lemma 5.2. Let $\mathbb{E}\alpha = 1$ and let Ξ be a Poisson process of intensity $\lambda < 1$. Call G the event that for every $\xi \in \Xi$ there exists $L < \infty$ such that $\xi + Q(L)$ is Ξ -replete for ξ . Then $\mathbf{P}(G) = 1$.

Proof. On G^C there exists a center $\xi \in \Xi$ such that for each L there exists a set of centers Ξ_L agreeing with Ξ on $\xi + Q(L)$ satisfying

$$\mathcal{L}[\psi_{\xi_L}^{-1}(\xi) \cap (\xi + Q(L))] < \alpha_\xi. \quad (14)$$

Define the appetite for the set of centers Ξ_L as in Theorem 1.4: by $\hat{\alpha}_n(\xi)$ for all $\xi \in \Xi_L$. Since $\Xi_L \Rightarrow \Xi$, we have that $\psi_L \rightarrow \psi_\Xi$, \mathbb{P} -as., by Theorem 1.4. Furthermore, Lemma 5.1 applies to ξ (by the subcritical phase in Theorem 1.2), so almost surely for L large enough, ξ is sated in each ψ_L and the radii $R_{\psi_L}(\xi)$ are bounded as $L \rightarrow \infty$. Thus G^C is null a.s. \square

Proposition 5.1. Let $\mathbb{E}\alpha = 1$ and let Ξ be a Poisson process with intensity $\lambda < 1$. For all $\varepsilon > 0$ there exists M such that

$$\mathcal{E} \{ \xi \in \Xi \cap Q(M) : Q(M) \text{ is not } [\Xi]\text{-replete for } \xi \} < \varepsilon(2M)^d.$$

Proof. For $A \subset \mathbb{R}^d$, call $\Xi^L(A)$ the number of $\xi \in \Xi \cap A$ such that $\xi + Q(L)$ is not Ξ -replete for ξ . Lemma 5.2 and the monotone convergence theorem imply that $\mathbf{E}\Xi^L(Q(1)) \rightarrow 0$ when $L \rightarrow \infty$. Thus we can choose $L < \infty$ so that the translation invariant point process Ξ^L has intensity less than $\varepsilon/2$. Note that for $M > L$ and $\xi \in \Xi \cap Q(M - L)$, if $Q(M)$ is not Ξ -replete for ξ , then $\xi \in \Xi^L$. Therefore

$$\begin{aligned} & \mathbf{E}\#\{\xi \in \Xi \cap Q(M) : Q(M) \text{ is not } \Xi\text{-replete for } \xi\} \\ & (\varepsilon/2)(2M - 2L)^d + (2M)^d - (2M - 2L)^d; \end{aligned}$$

which is smaller than $\varepsilon(2M)^d$ for large enough M . \square

Theorem 5.1. Assume α has mean 1 and variance σ^2 and let Ξ be a Poisson process with intensity $\lambda < 1$. Then there exists $c > 0$ such that, for all $r > 0$,

$$\mathcal{P}[\text{there exists } \xi \in \Xi \cap B(0, 1) \text{ such that } R(\xi) > r] < cr^{-d}.$$

Proof. Fix $\lambda < 1$ and note that if

$$\text{there exists } y \in B(0, r) \text{ with } \psi(y) \notin B(0, 2r + 1) \quad (15)$$

then $R(\xi) < r + 1$ for all $\xi \in \Xi \cap B(0, 1)$. Otherwise, $|y - \xi| < r + 1$ and $|y - \psi(y)| > r + 1$ and (y, ξ) would be unstable. Then it suffices to study

the tail of the complementary event in (15). Let $\varepsilon = (1 - \lambda)/(10^d 2^d)$ and let $M = M(\lambda, \varepsilon)$ as in Lemma 5.1.

Now, for all $r > 0$ we are going to place disjoint copies of $Q(M)$ in the shell $B(0, 2r + 1) \setminus B(0, r)$. For $z \in \mathbb{Z}^d$ let $Q_z = Q(M) + 2Mz$ and define

$$W_z = \{\xi \in \Xi \cap Q_z : Q_z \text{ is not } \Xi\text{-replete for } \xi\}.$$

Let

$$I = I(r) = \{z \in \mathbb{Z}^d : Q_z \subset B(0, 2r + 1) \setminus B(0, r)\}$$

be the index set of the cubes completely contained in the shell and let

$$S = S(r) = [B(0, 2r + 1) \setminus B(0, r)] \setminus \bigcup_{z \in I} Q_z \quad (16)$$

be the part of the shell which is uncovered by the cubes Q_z with $z \in I$. Consider the following events:

$$\begin{aligned} E &= \left\{ \sum_{\xi \in B(0, r)} \alpha_\xi < (\lambda + \varepsilon) \pi_d r^d \right\} \\ F &= \left\{ \sum_{\xi \in S} \alpha_\xi < \varepsilon \pi_d r^d \right\} \\ G &= \left\{ \sum_{z \in I} \sum_{\xi \in W_z} \alpha_\xi < 4^d \varepsilon 2^d \pi_d r^d \right\}. \end{aligned}$$

If E occurs, we have that

$$\begin{aligned} \mathcal{L}\{y \in B(0, r) : \psi(y) \notin B(0, r)\} &\geq \pi_d r^d - \sum_{\xi \in B(0, r)} \alpha_\xi \\ &\geq \pi_d r^d - (\lambda + \varepsilon) \pi_d r^d \\ &\geq 9^d \varepsilon 2^d \pi_d r^d, \end{aligned}$$

where the third inequality is due to the ε chosen. If F occurs, we have that

$$\mathcal{L}\{y \in B(0, r) : \psi(y) \in S\} < \sum_{\xi \in S} \alpha_\xi < \varepsilon \pi_d r^d.$$

If G occurs, by Definition 5.1, we have

$$\mathcal{L} \left\{ y \in B(0, r) : \psi(y) \in \bigcup_{z \in I} Q_z \right\} \leq \sum_{z \in I} \sum_{\xi \in W_z} \alpha_\xi < 4^d \varepsilon 2^d \pi_d r^d.$$

Since $B(0, 2r+1) = S \cup B(0, r) \cup \bigcup_{z \in I} Q_z$, we have that if the event $E \cap F \cap G$ occurs, then

$$\mathcal{L}\{y \in B(0, r) : \psi(y) \notin B(0, 2r+1)\} \geq (9^d 2^d - 4^d 2^d - 1) \varepsilon \pi_d r^d > 0.$$

Therefore, when E , F and G occur, the event in (15) occurs too. So it is enough to analyze the decay of $\mathcal{P}[E^C]$, $\mathcal{P}[F^C]$ and $\mathcal{P}[G^C]$. We start with G^C .

Note that $\#I$ is bounded by $\pi_d [(2r+1)^d - r^d] / (2M)^d < \pi_d 3^d r^d / (2M)^d$. Also, $\mathcal{E}(\#W_z)$ is bounded by $\varepsilon (2M)^d$ because of Proposition 5.1. With this considerations, we define a Poisson r.v. N having intensity $\pi_d 3^d r^d \varepsilon$. Define $K = \pi_d (3^d r^d \varepsilon + r^{d/2+\delta})$, for some small $\delta > 0$. Call $\alpha'_k = \alpha_k - 1$ for $k = 1, \dots, K$ and $a = 4^d \varepsilon 2^d \pi_d r^d - \lceil K \rceil$.

$$\begin{aligned} \mathcal{P}[G^C] &= \mathcal{P} \left[\sum_{z \in I} \sum_{\xi \in W_z} \alpha_\xi \geq 4^d \varepsilon 2^d \pi_d r^d \right] \\ &\leq \mathcal{P} \left[\sum_{i=1}^N \alpha_i \geq 4^d \varepsilon 2^d \pi_d r^d \mid N \leq K \right] \mathcal{P}[N \leq K] + \mathcal{P}[N > K] \\ &\leq \mathcal{P} \left[\sum_{i=1}^{\lceil K \rceil} \alpha'_i \geq a \right] + \mathcal{P}[N > K] \end{aligned}$$

The second term in the last inequality can be bounded by $e^{-c_3 r^{2\delta}}$ for some constant $c_3 = c_3(\lambda, d) > 0$, and the first term can be bounded using the Nagaev inequality in Proposition 3.1:

$$\begin{aligned} \mathcal{P} \left[\sum_{i=1}^{\lceil K \rceil} \alpha'_i > a \right] &\leq \frac{4 \lceil K \rceil A_2^+}{a^2} + \exp \left\{ \frac{-8a^2}{\lceil K \rceil e^2 \sigma^2} \right\} \\ &\leq c_1 r^{-d} + e^{-c_2 r^d}. \end{aligned}$$

Now we consider $\mathcal{P}[E^C]$. Let N' be a Poisson r.v. with mean $\lambda \pi_d r^d$. Call $\alpha'_i = \alpha_i - 1$; $a = (\lambda + \varepsilon) \pi_d r^d - K$ where $K = \lambda \pi_d (r^d + r^{d/2+\delta})$ for some small

$\delta > 0$. Note that a is positive for $r > (\varepsilon\pi_d)^{-1/d}$. Then

$$\begin{aligned}\mathcal{P}[E^C] &= \mathcal{P}\left[\sum_{\xi \in B(0,r)} \alpha_\xi \geq (\lambda + \varepsilon)\pi_d r^d\right] \\ &\leq \mathcal{P}\left[\sum_{i=1}^{N'} \alpha_i \geq (\lambda + \varepsilon)\pi_d r^d \mid N' \leq K\right] \mathcal{P}[N' \leq K] + \mathcal{P}[N' > K] \\ &\leq \mathcal{P}\left[\sum_{i=1}^K \alpha'_i \geq a\right] + \mathcal{P}[N' > K].\end{aligned}$$

By the Chernoff inequality (1), the second term is bounded by $e^{-C_1 r^{2\delta}}$, where $C_1 = C_1(\lambda, d) > 0$. By Proposition 3.1, the first term can be bounded as follows:

$$\begin{aligned}\mathcal{P}\left[\sum_{i=1}^K \alpha'_i \geq a\right] &\leq \frac{4[K]A_2^+}{a^2} + \exp\left\{\frac{-8a^2}{[K]e^2\sigma^2}\right\} \\ &\leq C_2 r^{-d} + e^{-C_3 r^d}.\end{aligned}$$

for some positive constants C_1 and C_2 . Thus, $\mathcal{P}[E^C]$ and $\mathcal{P}[G^C]$ decay as r^{-d} . A similar procedure will show that $\mathcal{P}[F^C]$ has the same polynomial decay, which proves the theorem. \square

Proof of Theorem 2.3. Let Y be the number of centers ξ in $\Xi \cap B(0, 1)$ such that $R(\xi) > r$. Let $u = r^{d/2}$. Note that

$$\begin{aligned}\mathcal{E}Y &= \mathcal{E}YI\{0 < Y \leq u\} + \mathcal{E}YI\{Y > u\} \\ &\leq u\mathcal{P}[Y > 0] + \mathcal{E}[\Pi \cap B(0, 1)]I\{\Pi \cap B(0, 1) > r\} \\ &\leq Cr^{-d/2} + \mathcal{E}[(\Pi \cap B(0, 1))^2]/u \\ &\leq Cr^{-d/2} + C_1 r^{-d/2}\end{aligned}$$

where the second inequality is obtained because, by Theorem 5.1, $\mathcal{P}[Y > 0]$ is bounded by Cr^{-d} , for some $C > 0$. The second term is bounded by $\mathcal{E}[(\Pi \cap B(0, 1))^2]/u$. By a property of the Palm process we have that $\mathcal{E}Y = \lambda\pi_d\mathcal{P}^*[R^* > r]$. Finally, after applying (5), we obtain the result. \square

6 Acknowledgments

The author is grateful to Serguei Popov, his Ph.D. advisor, for his guidance in the development of this work. He thanks the anonymous referee for the

very useful suggestions and the commentator “biosdev” on a post of the author’s blog (<http://wp.me/sm7GZ-513>). He also thanks Óscar Ocampo for some valuable insights. Finally, he acknowledges the financial support of CAPES during the development of his research.

References

- [1] DIAZ D. (2009). Algumas propriedades de alocações para o processo pontual de Poisson. Ph.D. Thesis, IME-USP.
- [2] DIAZ D. (2009). Percolation for the stable marriage of Poisson and Lebesgue with random appetites. Submitted.
- [3] GALE D., SHAPLEY L. (1962). College admission and stability of marriage. *Am. Math. Month.* **69** 9–15.
- [4] HOFFMAN C., HOLROYD A., PERES Y. (2005). A stable marriage of Poisson and Lebesgue. *Ann. Probab.* **34**(4) 1241–1272.
- [5] HOFFMAN C., HOLROYD A., PERES Y. Tail bounds for the stable marriage of Poisson and Lebesgue. *Can. Jnl. Math* Accepted to publication
- [6] HOLROYD A., PEMANTLE R., PERES Y., SCHRAMM O. Poisson Matching. Preprint.
- [7] MEESTER R., ROY R. (1996) *Continuum Percolation*. Cambridge University Press.
- [8] NAGAEV S. V. (1979) Large deviations of sums of independent random variables. *Ann. Probab.* **7**(5) 745–789.
- [9] THORISSON H. (1996) Transforming random elements and shifting random fields. *Ann. Probab.* **24** 2057–2064.