F tests for the strip-split plot design

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Abstract

In this article we present the structure of the F tests, the variance components and the approximate degrees of freedom for each of the eight possible mixed models of the strip-split plot design. We present an example to illustrate the model and compare it to more traditional settings like a three-way factorial design and a split-split plot model.

Key words: Experimental design, mixed models.

1 Introduction and Method

There are many opportunities in which a researcher needs to know the behavior of a factor in relation to one and/or two additional factors. When this happens a factorial design is usually considered, which is due, in part, to the great development reached by this type of model.

Instead, we study here the strip-split-plot design; i.e., an extension of strip-block designs such that each plot on the intersection is subdivided into subplots to insert a third factor. This new factor will be more precise on its measurement due to its high number of observations and interactions; which is the more important feature of the design.

We do not claim originality on the invention of this model. On the contrary, Gomez and Gomez (1984) described it, as well as Zimmermann (2004, 2014) did. They also described the F tests when the effects are fixed. Nonetheless, after an intensive search, we could not find on the literature those same F tests for the strip-split plot design with mixed effects. For instance, Montgomery (2012) calls strip-split-plots what is known in most of the remaining literature as strip blocks, therefore his analysis is developed for this latter case, and again only for the fixed effects model. The omission is understandable taking into account that his work is mainly focused on industrial applications; not agriculture, where this model could be more useful.

Other authors, like Cochran and Cox (1992) talk about strip-split plots and strip blocks, but do not talk about strip-split plot designs. Kuehl (1999) opens the possibility to a third factor for experiments with sub-subplots, but does not go beyond this point. McIntosh (1983) introduces analyses for combined experiments, but does not touch strip-split plot designs. Finally, Saavedra (2000) works with combined experiments in split plots and even works with sub-subplots, but she does not touch either strip-split plot designs.

Thus, there is a gap on the literature and no current monographs, books or papers, to our knowledge, seem to cover it. We intend to fill that gap here presenting for the first time the

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development of such F tests for all possible mixed models. On a sequel, we will consider the contrasts for this design and construct their variances and variance estimators, again for every case of the mixed effects model.

To determine the variance components and the ANOVA, we use a method explained by Searle et. al. (1992). The design is completely randomized so that it makes sense to implement F tests. The mathematical model is given by

$$y_{hijk} = m + R_h + A_i + e_{A_{hi}} + B_j + e_{B_{hj}} + AB_{ij} + e_{AB_{hij}} + C_k + AC_{ik} + BC_{jk} + ABC_{ijk} + e_t,$$

where m is the general mean, R_h is the h-th random block effect ($h = 1, \ldots, r$), A_i is the i-th horizontal strip effect ($i = 1, \ldots, a$), B_j is the j-th vertical strip effect ($j = 1, \ldots, b$) and C_k is the k-th effect of the subplot of A and B ($k = 1, \ldots, c$). So y_{hijk} represents the observation of the i-th level of A, the j-th level of B, the k-th level of C on the block h. The errors $e_{A_{hi}}$, $e_{B_{hj}}$, $e_{AB_{hij}}$ and $e_{t_{hijk}}$ are normally distributed with mean zero and variance $\sigma^2_{e_A}$, $\sigma^2_{e_B}$, $\sigma^2_{e_{AB}}$ and $\sigma^2_{e_t}$, respectively. Since the blocks are random, we will assume $R \sim N(0, \sigma^2_R)$.

The analysis is done according to the scheme on Table 1, where *df* stands for degrees of freedom and SS stands for the sum of squares of the respective variation source. It is worth mentioning here that Díaz (2004) presents the sums of squares and the covariance matrices for all the mixed models as Kronecker products, but those are omitted here to save space.

Table 1: Sums of squares and Degrees of freedom

Source	*df*	SS
R	$r-1$	$abc\sum_{h=1}^{r}(\overline{y}_{h...} - \overline{y}_{....})^2$
A	$a-1$	$bcr\sum_{i=1}^{a}(\overline{y}_{.i..} - \overline{y}_{....})^2$
e_A	$(r-1)(a-1)$	$bc\sum_{i=1}^{a}\sum_{h=1}^{r}(\overline{y}_{hi..} - \overline{y}_{h...} - \overline{y}_{.i..} + \overline{y}_{....})^2$
B	$b-1$	$acr\sum_{j=1}^{b}(\overline{y}_{..j.} - \overline{y}_{....})^2$
e_B	$(r-1)(b-1)$	$ac\sum_{j=1}^{b}\sum_{h=1}^{r}(\overline{y}_{h.j.} - \overline{y}_{h...} - \overline{y}_{..j.} + \overline{y}_{....})^2$
AB	$(a-1)(b-1)$	$cr\sum_{i=1}^{a}\sum_{j=1}^{b}(\overline{y}_{.ij.} - \overline{y}_{.i..} - \overline{y}_{..j.} + \overline{y}_{....})^2$
e_{AB}	$(a-1)(b-1)(r-1)$	$c\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{h=1}^{r}(\overline{y}_{hij.} - \overline{y}_{hi..} - \overline{y}_{h.j.} - \overline{y}_{.ij.} + \overline{y}_{h...} + \overline{y}_{.i..} + \overline{y}_{..j.} - \overline{y}_{....})^2$
C	$c-1$	$abr\sum_{k=1}^{c}(\overline{y}_{...k} - \overline{y}_{....})^2$
AC	$(a-1)(c-1)$	$br\sum_{i=1}^{a}\sum_{k=1}^{c}(\overline{y}_{.i.k} - \overline{y}_{.i..} - \overline{y}_{...k} + \overline{y}_{....})^2$
BC	$(b-1)(c-1)$	$ar\sum_{j=1}^{b}\sum_{k=1}^{c}(\overline{y}_{..jk} - \overline{y}_{..j.} - \overline{y}_{...k} + \overline{y}_{....})^2$
ABC	$(a-1)(b-1)(c-1)$	$r\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}(\overline{y}_{.ijk} - \overline{y}_{.i.k} - \overline{y}_{..jk} + \overline{y}_{...k} - \overline{y}_{.ij.} + \overline{y}_{.i..} + \overline{y}_{..j.} - \overline{y}_{....})^2$
e_t	$ab(c-1)(r-1)$	$\sum_{h=1}^{r}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}(y_{hijk} - \overline{y}_{.ijk} - \overline{y}_{hij.} + \overline{y}_{.ij.})^2$

2 Expected mean squares

To illustrate how to obtain the expected mean squares $E(MS)$ we will show the process for the random blocks R (For the remaining cases, since the procedure is similar, we will only present the final value without the respective development): First, take SS_R in Table 1 and calculate its expected value:

$$E(SS_R) = abc\sum_{h=1}^{r} E(R_h - \overline{R}_{.} + \overline{e}_{A_{h.}} - \overline{e}_{A_{..}} + \overline{e}_{B_{h.}} - \overline{e}_{B_{..}} + \overline{e}_{AB_{h..}} - \overline{e}_{AB_{...}} + \overline{e}_{t_{h...}} - \overline{e}_{t_{....}})^2,$$

since the product of errors and factors is always zero under expectation, $E(SS_R)$ equals

$$abc\sum_{h=1}^{r} E(R_h - \overline{R}_{.})^2 + abc\sum_{h=1}^{r} E(\overline{e}_{A_{h.}} - \overline{e}_{A_{..}} + \overline{e}_{B_{h.}} - \overline{e}_{B_{..}} + \overline{e}_{AB_{h..}} - \overline{e}_{AB_{...}} + \overline{e}_{t_{h...}} - \overline{e}_{t_{....}})^2,$$

and since the errors are independent between themselves,

$$\begin{aligned} E(SS_R) = abc\sum_{h=1}^{r} E(R_h - \overline{R}_{.})^2 + abc\sum_{h=1}^{r} E(\overline{e}_{A_{h.}} - \overline{e}_{A_{..}})^2 + abc\sum_{h=1}^{r} E(\overline{e}_{B_{h.}} - \overline{e}_{B_{..}})^2 \\ + abc\sum_{h=1}^{r} E(\overline{e}_{AB_{h..}} - \overline{e}_{AB_{...}})^2 + abc\sum_{h=1}^{r} E(\overline{e}_{t_{h...}} - \overline{e}_{t_{....}})^2. \end{aligned}$$

Therefore, taking into account that $\sigma_e^2 = E(e^2) - E^2(e)$, and that $E(e) = 0$ for every error in the model,

$$\begin{aligned} E(SS_R) = abc\sum_{h=1}^{r} E(R_h - \overline{R}_{.})^2 + abc\frac{(r-1)\sigma^2_{e_A}}{a} \\ + abc\frac{(r-1)\sigma^2_{e_B}}{b} + abc\frac{(r-1)\sigma^2_{e_{AB}}}{ab} + abc\frac{(r-1)\sigma^2_{e_t}}{abc}. \end{aligned}$$

Then, taking $E(SS_R)$ and dividing it by its *df*, we obtain the expected mean square for fixed blocks:

$$E(MS_R) = \frac{abc}{r-1}\sum_{h=1}^{r} E(R_h - \overline{R}_{.})^2 + bc\sigma^2_{e_A} + ac\sigma^2_{e_B} + c\sigma^2_{e_{AB}} + \sigma^2_{e_t}. \tag{1}$$

Now, for the more interesting case of random blocks, we get:

$$E(MS_R) = abc\sigma^2_R + bc\sigma^2_{e_A} + ac\sigma^2_{e_B} + c\sigma^2_{e_AB} + \sigma^2_{e_t}.$$

Note that $E(MS_R)$ will remain unchanged regardless the model we are considering. This is also true for the expectation of the mean square of each error involved. So we mention these here and will omit them in the particular description of the $E(MS)$'s for each model:

$$\begin{aligned} E(MS_{e_A}) &= bc\sigma^2_{e_A} + c\sigma^2_{e_{AB}} + \sigma^2_{e_t}, \\ E(MS_{e_B}) &= ac\sigma^2_{e_B} + c\sigma^2_{e_{AB}} + \sigma^2_{e_t}, \\ E(MS_{e_{AB}}) &= c\sigma^2_{e_{AB}} + \sigma^2_{e_t}, \\ E(MS_{e_t}) &= \sigma^2_{e_t}. \end{aligned}$$

Finally, note also that every interaction involving a random effect will be random. So in the following subsections, to avoid confusion, we present explicitly all the $E(MS)$'s for every model.

2.1 Expected mean squares for the fixed effects model

When the effects are fixed (constant), by definition it is sufficient to suppress the expectation operator of the mean squares considered. Thus we get:

$$
\begin{aligned}
E(MS_A) &= \frac{bcr}{a-1}\sum_{i=1}^{a}(A_i - \overline{A}_{.} + \overline{AB}_{i.} - \overline{AB}_{..} + \overline{AC}_{i.} - \overline{AC}_{..} + \overline{ABC}_{i..} - \overline{ABC}_{...})^2 \\
&\quad + bc\sigma^2_{e_A} + c\sigma^2_{e_{AB}} + \sigma^2_{e_t}, \\
E(MS_B) &= \frac{acr}{b-1}\sum_{j=1}^{b}(B_j - \overline{B}_{.} + \overline{AB}_{.j} - \overline{AB}_{..} + \overline{BC}_{j.} - \overline{BC}_{..} + \overline{ABC}_{.j.} - \overline{ABC}_{...})^2 \\
&\quad + ac\sigma^2_{e_B} + c\sigma^2_{e_{AB}} + \sigma^2_{e_t}, \\
E(MS_{AB}) &= \frac{cr}{(a-1)(b-1)}\sum_{i=1}^{a}\sum_{j=1}^{b}(AB_{ij} - \overline{AB}_{.j} - \overline{AB}_{i.} + \overline{AB}_{..} \\
&\quad + \overline{ABC}_{ij.} - \overline{ABC}_{.j.} - \overline{ABC}_{i..} + \overline{ABC}_{...})^2 + c\sigma^2_{e_{AB}} + \sigma^2_{e_t}, \\
E(MS_C) &= \frac{abr}{c-1}\sum_{k=1}^{c}(C_k - \overline{C}_{.} + \overline{AC}_{.k} - \overline{AC}_{..} + \overline{BC}_{.k} - \overline{BC}_{..} + \overline{ABC}_{..k} - \overline{ABC}_{...})^2 + \sigma^2_{e_t}, \\
E(MS_{AC}) &= \frac{br}{(a-1)(c-1)}\sum_{i=1}^{a}\sum_{k=1}^{c}(AC_{ik} - \overline{AC}_{.k} - \overline{AC}_{i.} + \overline{AC}_{..} \\
&\quad + \overline{ABC}_{i.k} - \overline{ABC}_{..k} - \overline{ABC}_{i..} + \overline{ABC}_{...})^2 + \sigma^2_{e_t}, \\
E(MS_{BC}) &= \frac{ar}{(b-1)(c-1)}\sum_{j=1}^{b}\sum_{k=1}^{c}(BC_{jk} - \overline{BC}_{.k} - \overline{BC}_{j.} + \overline{BC}_{..} \\
&\quad + \overline{ABC}_{.jk} - \overline{ABC}_{..k} - \overline{ABC}_{.j.} + \overline{ABC}_{...})^2 + \sigma^2_{e_t}, \\
E(MS_{ABC}) &= \frac{r}{(a-1)(b-1)(c-1)}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}(ABC_{ijk} - \overline{ABC}_{.jk} \\
&\quad - \overline{ABC}_{i.k} + \overline{ABC}_{..k} - \overline{ABC}_{ij.} + \overline{ABC}_{.j.} + \overline{ABC}_{i..} - \overline{ABC}_{...})^2 + \sigma^2_{e_t}.
\end{aligned}
$$

2.2 Expected mean squares for the random effects model

When a factor, say A, has random effects, we will assume that the effects of A have distribution $N(0, \sigma^2_A)$. Then for the random effects model, the effects of A, B and C will be random, independent, and normally distributed with mean 0 and variance σ^2_A, σ^2_B and σ^2_C, respectively. The interactions AB, AC, BC and ABC will have normal distribution with mean 0 and variance σ^2_{AB}, σ^2_{AC}, σ^2_{BC} and σ^2_{ABC}, respectively. We also assume that the effects are independent between them. So we get:

$$
\begin{aligned}
E(MS_A) &= bcr\sigma_A^2 + bc\sigma_{e_A}^2 + cr\sigma_{AB}^2 + c\sigma_{e_{AB}}^2, +br\sigma_{AC}^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_B) &= acr\sigma_B^2 + ac\sigma_{e_B}^2 + cr\sigma_{AB}^2 + c\sigma_{e_{AB}}^2, +ar\sigma_{BC}^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_{AB}) &= cr\sigma_{AB}^2 + c\sigma_{e_{AB}}^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_C) &= abr\sigma_C^2 + br\sigma_{AC}^2 + ar\sigma_{BC}^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_{AC}) &= br\sigma_{AC}^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_{BC}) &= ar\sigma_{BC}^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_{ABC}) &= r\sigma_{ABC}^2 + \sigma_{e_t}^2.
\end{aligned}
$$

2.3 Expected mean squares when only A is fixed

In this case, B and C will be random with variances σ_B^2 and σ_C^2, respectively. Also, AB, AC, BC and ABC will be random with variances σ_{AB}^2, σ_{AC}^2, σ_{BC}^2 and σ_{ABC}^2, respectively. So we obtain:

$$
\begin{aligned}
E(MS_A) &= \frac{bcr}{a-1}\sum_{i=1}^{a}(A_i - \overline{A}_.)^2 + bc\sigma_{e_A}^2 + cr\sigma_{AB}^2 + c\sigma_{e_{AB}}^2 + br\sigma_{AC}^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_B) &= acr\sigma_B^2 + ac\sigma_{e_B}^2 + cr\sigma_{AB}^2 + c\sigma_{e_{AB}}^2 + ar\sigma_{BC}^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_{AB}) &= cr\sigma_{AB}^2 + c\sigma_{e_{AB}}^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_C) &= abr\sigma_C^2 + br\sigma_{AC}^2 + ar\sigma_{BC}^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_{AC}) &= br\sigma_{AC}^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_{BC}) &= ar\sigma_{BC}^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_{ABC}) &= r\sigma_{ABC}^2 + \sigma_{e_t}^2.
\end{aligned}
$$

2.4 Expected mean squares when only B is fixed

Here we have that A and C will be random with variances σ_A^2 and σ_C^2, respectively. Also, AB, AC, BC and ABC will be random with variances σ_{AB}^2, σ_{AC}^2, σ_{BC}^2 and σ_{ABC}^2, respectively. So we obtain:

$$
\begin{aligned}
E(MS_A) &= bcr\sigma_A^2 + bc\sigma_{e_A}^2 + cr\sigma_{AB}^2 + c\sigma_{e_{AB}}^2 + br\sigma_{AC}^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_B) &= \frac{acr}{b-1}\sum_{j=1}^{b}(B_j - \overline{B}_.)^2 + ac\sigma_{e_B}^2 + cr\sigma_{AB}^2 + c\sigma_{e_{AB}}^2 + ar\sigma_{BC}^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_{AB}) &= cr\sigma_{AB}^2 + c\sigma_{e_{AB}}^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_C) &= abr\sigma_C^2 + br\sigma_{AC}^2 + ar\sigma_{BC}^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_{AC}) &= br\sigma_{AC}^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_{BC}) &= ar\sigma_{BC}^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_{ABC}) &= r\sigma_{ABC}^2 + \sigma_{e_t}^2.
\end{aligned}
$$

2.5 Expected mean squares when only C is fixed

In this case, A and B are random with variances σ_A^2 and σ_B^2, respectively. Also, AB, AC, BC and ABC will be random with variances σ_{AB}^2, σ_{AC}^2, σ_{BC}^2 and σ_{ABC}^2, respectively. So we obtain:

$$\begin{aligned}
E(MS_A) &= bcr\sigma_A^2 + bc\sigma_{e_A}^2 + cr\sigma_{AB}^2 + c\sigma_{e_{AB}}^2 + br\sigma_{AC}^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_B) &= acr\sigma_B^2 + ac\sigma_{e_B}^2 + cr\sigma_{AB}^2 + c\sigma_{e_{AB}}^2 + ar\sigma_{BC}^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_{AB}) &= cr\sigma_{AB}^2 + c\sigma_{e_{AB}}^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_C) &= \frac{abr}{c-1}\sum_{k=1}^{c}(C_k - \overline{C}_.)^2 + br\sigma_{AC}^2 + ar\sigma_{BC}^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_{AC}) &= br\sigma_{AC}^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_{BC}) &= ar\sigma_{BC}^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_{ABC}) &= r\sigma_{ABC}^2 + \sigma_{e_t}^2.
\end{aligned}$$

2.6 Expected mean squares when only A is random

In this case, A, AB, AC and ABC are random with variance σ_A^2, σ_{AB}^2, σ_{AC}^2 and σ_{ABC}^2, respectively. Thus, we get:

$$\begin{aligned}
E(MS_A) &= bcr\sigma_A^2 + bc\sigma_{e_A}^2 + cr\sigma_{AB}^2 + c\sigma_{e_{AB}}^2 + br\sigma_{AC}^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_B) &= \frac{acr}{b-1}\sum_{j=1}^{b}(B_j - \overline{B}_. + \overline{BC}_{j.} - \overline{BC}_{..})^2 + ac\sigma_{e_B}^2 + cr\sigma_{AB}^2 + c\sigma_{e_{AB}}^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_{AB}) &= cr\sigma_{AB}^2 + c\sigma_{e_{AB}}^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_C) &= \frac{abr}{c-1}\sum_{k=1}^{c}(C_k - \overline{C}_. + \overline{BC}_{.k} - \overline{BC}_{..})^2 + br\sigma_{AC}^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_{AC}) &= br\sigma_{AC}^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_{BC}) &= \frac{ar}{(b-1)(c-1)}\sum_{j=1}^{b}\sum_{k=1}^{c}(BC_{jk} - \overline{BC}_{.k} - \overline{BC}_{j.} + \overline{BC}_{..})^2 + r\sigma_{ABC}^2 + \sigma_{e_t}^2,\\
E(MS_{ABC}) &= r\sigma_{ABC}^2 + \sigma_{e_t}^2.
\end{aligned}$$

2.7 Expected mean squares when only B is random

In this case, B, AB, BC and ABC are random with variance σ_B^2, σ_{AB}^2, σ_{BC}^2 and σ_{ABC}^2, respectively. Thus, we get:

$$
\begin{aligned}
E(MS_A) &= \frac{bcr}{a-1}\sum_{i=1}^{a}(A_i - \overline{A}_{\cdot} + \overline{AC}_{i\cdot} - \overline{AC}_{\cdot\cdot})^2 + bc\sigma^2_{e_A} + cr\sigma^2_{AB} + c\sigma^2_{e_{AB}} + r\sigma^2_{ABC} + \sigma^2_{e_t},\\
E(MS_B) &= acr\sigma^2_B + ac\sigma^2_{e_B} + cr\sigma^2_{AB} + c\sigma^2_{e_{AB}} + ar\sigma^2_{BC} + r\sigma^2_{ABC} + \sigma^2_{e_t},\\
E(MS_{AB}) &= cr\sigma^2_{AB} + c\sigma^2_{e_{AB}} + r\sigma^2_{ABC} + \sigma^2_{e_t},\\
E(MS_C) &= \frac{abr}{c-1}\sum_{k=1}^{c}(C_k - \overline{C}_{\cdot} + \overline{AC}_{\cdot k} - \overline{AC}_{\cdot\cdot})^2 + ar\sigma^2_{BC} + r\sigma^2_{ABC} + \sigma^2_{e_t},\\
E(MS_{AC}) &= \frac{br}{(a-1)(c-1)}\sum_{i=1}^{a}\sum_{k=1}^{c}(AC_{ik} - \overline{AC}_{\cdot k} - \overline{AC}_{i\cdot} + \overline{AC}_{\cdot\cdot})^2 + r\sigma^2_{ABC} + \sigma^2_{e_t},\\
E(MS_{BC}) &= ar\sigma^2_{BC} + r\sigma^2_{ABC} + \sigma^2_{e_t},\\
E(MS_{ABC}) &= r\sigma^2_{ABC} + \sigma^2_{e_t}.
\end{aligned}
$$

2.8 Expected mean squares when only C is random

Here C, AC, BC and ABC are random with variance σ^2_C, σ^2_{AC}, σ^2_{BC} and σ^2_{ABC}, respectively. Thus, we get:

$$
\begin{aligned}
E(MS_A) &= \frac{bcr}{a-1}\sum_{i=1}^{a}(A_i - \overline{A}_{\cdot} + \overline{AB}_{i\cdot} - \overline{AB}_{\cdot\cdot})^2 + bc\sigma^2_{e_A} + c\sigma^2_{e_{AB}} + br\sigma^2_{AC} + r\sigma^2_{ABC} + \sigma^2_{e_t},\\
E(MS_B) &= \frac{acr}{b-1}\sum_{j=1}^{b}(B_j - \overline{B}_{\cdot} + \overline{AB}_{\cdot j} - \overline{AB}_{\cdot\cdot})^2 + ac\sigma^2_{e_B} + c\sigma^2_{e_{AB}} + ar\sigma^2_{BC} + r\sigma^2_{ABC} + \sigma^2_{e_t},\\
E(MS_{AB}) &= \frac{cr}{(a-1)(b-1)}\sum_{i=1}^{a}\sum_{j=1}^{b}(AB_{ij} - \overline{AB}_{\cdot j} - \overline{AB}_{i\cdot} + \overline{AB}_{\cdot\cdot})^2 + r\sigma^2_{ABC} + c\sigma^2_{e_{AB}} + \sigma^2_{e_t},\\
E(MS_C) &= abr\sigma^2_C + br\sigma^2_{AC} + ar\sigma^2_{BC} + r\sigma^2_{ABC} + \sigma^2_{e_t},\\
E(MS_{AC}) &= br\sigma^2_{AC} + r\sigma^2_{ABC} + \sigma^2_{e_t},\\
E(MS_{BC}) &= ar\sigma^2_{BC} + r\sigma^2_{ABC} + \sigma^2_{e_t},\\
E(MS_{ABC}) &= r\sigma^2_{ABC} + \sigma^2_{e_t}.
\end{aligned}
$$

3 F tests

In this section we present the F tests in tables. When required, we will specify the approximated *df* by means of the famous estimator developed by Satterthwaite (1946); for the cases in which the complex estimation is a function of two variance components we will use the estimator proposed by Ames and Webster (1991), which is a correction to Satterthwaite for this particular case. Let us start with the estimator by Satterthwaite:

If θ is variance which is a linear combination of m independent variances, i.e., if $\theta = \sum_{i=1}^{m} a_i\theta_i$, with estimator $\hat{\theta} = \sum_{i=1}^{m} a_i MS_i^2$, we say that $\hat{\theta}$ is a complex estimator of θ. Since for our case the coefficients $a_i = 1$, for $i = 1, \ldots, m$, we will omit them on what follows. For the cases in which the variance estimator is complex, Satterthwaite (1946) proposed the following estimator for the *df*:

$$\hat{f}_s = \frac{\left(\sum_{i=1}^{m} MS_i\right)^2}{\sum_{i=1}^{m} MS_i^2/n_i}, \tag{2}$$

where n_i are the *df* of the source of variation corresponding to i. This is so because $\frac{f\hat{\theta}}{\theta}$ can be approximated to a χ^2 with f degrees of freedom.

Ames and Webster (1991) consider this estimator unstable —and they are right— because of its denominator. Notice that on this denominator each variance estimator is first squared and then added. Since in the numerator, the terms are first added and the result is squared, $\hat{f}_s$ can be affected. Moreover, if the variance components are underestimated, there is the undesirable risk of overestimating the degrees of freedom. For these reasons they propose the following estimator:

When the variance θ is a function of two mean squares, θ_1 and θ_2, call $\phi_1 = 1$ and $\phi_2 = \theta_2/\theta_1$, and consider the class of estimators given by $\hat{\phi}_2 = rMS_2/MS_1$, where r is a constant, then we can approximate the *df* by

$$\hat{f}_{aw}(r) = \frac{\left(\sum_{i=1}^{2} \hat{\phi}_i\right)^2}{\sum_{i=1}^{2} \phi_i^2/n_i}. \tag{3}$$

Note that $\hat{f}_{aw}(1) = \hat{f}_s$ and that $\min(n_1, n_2) \leq \hat{f}_{aw}(r) \leq n_1 + n_2$. Thus, we can vary r in order to get better properties. For instance,

$$r^* = \frac{n_2}{n_2 - 2}\left(\frac{2(n_1 + n_2 - 2)}{n_1(n_2 - 4)} + 1\right)$$

minimizes the mean square of the error of $1/\hat{\phi}_2$. Also $r^* > 1$ and $\hat{f}_{aw}(r^*) < f_s$. In this paper, every time we calculate the Ames-Webster estimator (3), we will also calculate its respective value r^*. Using the Ames-Webster approach we have two possible estimations for every value of r. Then, if both of them are less than $\hat{f}_s$, it is advisable to use the larger one, since the smaller one usually has a negative bias.

With these tools at hand, we proceed to present the F test for every model. The first column in each of the tables will be the source of variation, the second one will tell us if the effects are random or fixed, the third one will be the corresponding F test and the last one will be the null hypothesis under consideration. When the effects are random, the null hypothesis will be that the corresponding variance of the source has 0 variance; when the effects are fixed, the null hypothesis will be that all effects are equal (to 0).

3.1 F tests when all effects are fixed

To construct the F tests in Table 2, we use the expected mean squares found in Subsection 2.1. Note that R will have the same structure for the F test, regardless of it being constant or random (although, of course, the hypothesis will change).

Using the Satterthwaite estimator in (2), we approximate the *df* for R as:

$$v_1 = \frac{(MS_R + MS_{e_{AB}})^2}{\frac{MS_R^2}{r-1} + \frac{MS_{e_{AB}}^2}{(r-1)(a-1)(b-1)}},$$

$$v_2 = \frac{(MS_{e_A} + MS_{e_B})^2}{\frac{MS_{e_A}^2}{(r-1)(a-1)} + \frac{MS_{e_B}^2}{(r-1)(b-1)}},$$

Table 2: F tests for the fixed effects model

Source	Effect	F	H_0
R	f	$\frac{MS_R+MS_{e_{AB}}}{MS_{e_A}+MS_{e_B}}$	$\sigma^2_R = 0$
A	f	$\frac{MS_A}{MS_{e_A}}$	$A_1 = A_2 = \cdots = A_a = 0$
e_A	r	$\frac{MS_{e_A}}{MS_{e_{AB}}}$	$\sigma^2_{e_A} = 0$
B	f	$\frac{MS_B}{MS_{e_B}}$	$B_1 = B_2 = \cdots = B_b = 0$
e_B	r	$\frac{MS_{e_B}}{MS_{e_{AB}}}$	$\sigma^2_{e_B} = 0$
AB	f	$\frac{MS_{AB}}{MS_{e_{AB}}}$	$(AB)_{ij} = 0, \forall i, \forall j.$
e_{AB}	r	$\frac{MS_{e_{AB}}}{MS_{e_t}}$	$\sigma^2_{e_{AB}} = 0$
C	f	$\frac{MS_C}{MS_{e_t}}$	$C_1 = C_2 = \cdots = C_c = 0$
AC	f	$\frac{MS_{AC}}{MS_{e_t}}$	$(AC)_{ik} = 0, \forall i, \forall k.$
BC	f	$\frac{MS_{BC}}{MS_{e_t}}$	$(BC)_{jk} = 0, \forall j, \forall k.$
ABC	f	$\frac{MS_{ABC}}{MS_{e_t}}$	$(ABC)_{ijk} = 0, \forall i, \forall j, \forall k.$
e_t	r	$-$	

where v_1 and v_2 are the *df* for the enumerator and the denominator, respectively.

When we adjust using the Ames-Webster estimator (3), we obtain two estimators for each case. First let us see the *df* for the numerator: Let $MS_1 = MS_R$ and $MS_2 = MS_{e_{AB}}$, then

$$p_1 = \frac{(r-1)(a-1)(b-1)}{(r-1)(a-1)(b-1)-2}\left(\frac{2[(r-1)(a-1)(b-1)+r-3]}{(r-1)[(r-1)(a-1)(b-1)-4]}+1\right),$$

$$\hat{f}_{aw}(p_1) = \frac{(1+p_1 MS_{e_{AB}}/MS_R)^2}{\frac{1}{r-1}+\frac{(p_1 MS_{e_{AB}}/MS_R)^2}{(r-1)(a-1)(b-1)}};$$

on the other hand, when $MS_1 = MS_{e_{AB}}$ and $MS_2 = MS_R$:

$$p_1^* = \frac{r-1}{r-3}\left(\frac{2[(r-1)(a-1)(b-1)+r-3]}{(r-1)(a-1)(b-1)(r-5)}+1\right),$$

$$\hat{f}_{aw}(p_1^*) = \frac{(1+p_1^* MS_R/MS_{e_{AB}})^2}{\frac{1}{(r-1)(a-1)(b-1)}+\frac{(p_1^* MS_R/MS_{e_{AB}})^2}{r-1}}.$$

Now, for the denominator, when $MS_1 = MS_{e_A}$ and $MS_2 = MS_{e_B}$ we have:

$$p_2 = \frac{(r-1)(b-1)}{(r-1)(b-1)-2}\left(\frac{2\{(r-1)[(a-1)+(b-1)]-2\}}{(r-1)(a-1)[(r-1)(b-1)-4]}+1\right),$$

$$\hat{f}_{aw}(p_2) = \frac{(1+p_2 MS_{e_B}/MS_{e_A})^2}{\frac{1}{(r-1)(a-1)}+\frac{(p_2 MS_{e_B}/M-3)^2}{(r-1)(b-1)}};$$

and when $MS_1 = MS_{e_B}$ and $MS_2 = MS_{e_A}$ we have:

$$p_2^* = \frac{(r-1)(a-1)}{(r-1)(a-1)-2}\left(\frac{2\{(r-1)[(a-1)+(b-1)]-2\}}{(r-1)(b-1)[(r-1)(a-1)-4]}+1\right),$$

$$\hat{f}_{aw}(p_2^*) = \frac{(1 + p_2^* MS_{e_A}/MS_{e_B})^2}{\frac{1}{(r-1)(b-1)} + \frac{(p_2^* MS_{e_A}/M-5)^2}{(r-1)(a-1)}}.$$

The estimators for the *df* of R will always be the same. For this reason they will be omitted on the tables to come.

3.2 F tests when all effects are random

When all effects are random, we construct the F tests on Table 3 based on the mean squares developed in Subsection 2.2.

Table 3: F tests for the random effects model

Source	Effect	F	H_0
R	r	$\frac{MS_R+MS_{e_{AB}}}{MS_{e_A}+MS_{e_B}}$	$\sigma_R^2 = 0$
A	r	$\frac{MS_A+MS_{e_{AB}}+MS_{ABC}}{MS_{e_A}+MS_{AB}+MS_{AC}}$	$\sigma_A^2 = 0$
e_A	r	$\frac{MS_{e_A}}{MS_{e_{AB}}}$	$\sigma_{AR}^2 = 0$
B	r	$\frac{MS_B+MS_{e_{AB}}+MS_{ABC}}{MS_{e_B}+MS_{AB}+MS_{BC}}$	$\sigma_B^2 = 0$
e_B	r	$\frac{MS_{e_B}}{MS_{e_{AB}}}$	$\sigma_{BR}^2 = 0$
AB	r	$\frac{MS_{AB}+MS_{e_t}}{MS_{e_{AB}}+MS_{ABC}}$	$\sigma_{AB}^2 = 0$
e_{AB}	r	$\frac{MS_{e_{AB}}}{MS_{e_t}}$	$\sigma_{e_{AB}}^2 = 0$
C	r	$\frac{MS_C+MS_{ABC}}{MS_{AC}+MS_{BC}}$	$\sigma_C^2 = 0$
AC	r	$\frac{MS_{AC}}{MS_{ABC}}$	$\sigma_{AC}^2 = 0$
BC	r	$\frac{MS_{BC}}{MS_{ABC}}$	$\sigma_{BC}^2 = 0$
ABC	r	$\frac{MS_{ABC}}{MS_{e_t}}$	$\sigma_{ABC}^2 = 0$
e_t	r	–	

Since the complex estimators for effects A and B in Table 3 have three variance components, we will use only (2) with them to find their approximate *df*. For the effects of A, the *df* in the numerator and the denominator v_1 and v_2, respectively, will be given by:

$$v_1 = \frac{(MS_A + MS_{e_{AB}} + MS_{ABC})^2}{\frac{MS_A^2}{a-1} + \frac{MS_{e_{AB}}^2}{(a-1)(b-1)(r-1)} + \frac{MS_{ABC}^2}{(a-1)(b-1)(c-1)}},$$
$$v_2 = \frac{(MS_{e_A} + MS_{AB} + MS_{AC})^2}{\frac{MS_{e_A}^2}{(r-1)(a-1)} + \frac{MS_{AB}^2}{(a-1)(b-1)} + \frac{MS_{AC}^2}{(a-1)(c-1)}}. \quad (4)$$

With B, the *df* of the F test will be for the numerator and denominator respectively:

$$v_1 = \frac{(MS_B + MS_{e_{AB}} + MS_{ABC})^2}{\frac{MS_B^2}{b-1} + \frac{MS_{e_{AB}}^2}{(a-1)(b-1)(r-1)} + \frac{MS_{ABC}^2}{(a-1)(b-1)(c-1)}},$$
$$v_2 = \frac{(MS_{e_B} + MS_{AB} + MS_{BC})^2}{\frac{MS_{e_B}^2}{(r-1)(b-1)} + \frac{MS_{AB}^2}{(a-1)(b-1)} + \frac{MS_{BC}^2}{(b-1)(c-1)}}. \quad (5)$$

The *df* for AB approximated by (2) will be respectively for the numerator and the denominator:

$$v_1 = \frac{(MS_{AB} + MS_{e_t})^2}{\frac{MS_{AB}^2}{(a-1)(b-1)} + \frac{MS_{e_t}^2}{ab(c-1)(r-1)}},$$
$$v_2 = \frac{(MS_{e_{AB}} + MS_{ABC})^2}{\frac{MS_{e_{AB}}^2}{(a-1)(b-1)(r-1)} + \frac{MS_{ABC}^2}{(a-1)(b-1)(c-1)}}. \tag{6}$$

Adjusting by means of (3), there are two possible estimators in each case. First, let us see the degrees of freedom in the numerator. Let $MS_1 = MS_{AB}$ and $MS_2 = MS_{e_t}$, then

$$p_1 = \frac{ab(c-1)(r-1)}{ab(c-1)(r-1)-2}\left(\frac{2[(a-1)(b-1)+ab(c-1)(r-1)-2]}{(a-1)(b-1)[ab(c-1)(r-1)-4]}+1\right),$$

$$\hat{f}_{aw}(p_1) = \frac{(1+p_1 MS_{e_t}/MS_{AB})^2}{\frac{1}{(a-1)(b-1)} + \frac{(p_1 MS_{e_t}/MS_{AB})^2}{ab(c-1)(r-1)}}; \tag{7}$$

on the other hand, when $MS_1 = MS_{e_t}$ and $MS_2 = MS_{AB}$:

$$p_1^* = \frac{(a-1)(b-1)}{(a-1)(b-1)-2}\left(\frac{2[(a-1)(b-1)+ab(c-1)(r-1)-2]}{ab(c-1)(r-1)[(a-1)(b-1)-4]}+1\right),$$

$$\hat{f}_{aw}(p_1^*) = \frac{(1+p_1^* MS_{AB}/MS_{e_t})^2}{\frac{1}{ab(c-1)(r-1)} + \frac{(p_1^* MS_{AB}/MS_{e_t})^2}{(a-1)(b-1)}}. \tag{8}$$

And for the denominator, when $MS_1 = MS_{e_{AB}}$ and $MS_2 = MS_{ABC}$ we have:

$$p_2 = \frac{(a-1)(b-1)(c-1)}{(a-1)(b-1)(c-1)-2}\left(\frac{2[(a-1)(b-1)(c+r-2)-2]}{(a-1)(b-1)(r-1)[(a-1)(b-1)(c-1)-4]}+1\right),$$

$$\hat{f}_{aw}(p_2) = \frac{(1+p_2 MS_{ABC}/MS_{e_{AB}})^2}{\frac{1}{(a-1)(b-1)(r-1)} + \frac{(p_2 MS_{ABC}/MS_{e_{AB}})^2}{(a-1)(b-1)(c-1)}}; \tag{9}$$

when $MS_1 = MS_{ABC}$ and $MS_2 = MS_{e_{AB}}$ we get:

$$p_2^* = \frac{(a-1)(b-1)(r-1)}{(a-1)(b-1)(r-1)-2}\left(\frac{2[(a-1)(b-1)(c+r-2)-2]}{(a-1)(b-1)(c-1)[(a-1)(b-1)(r-1)-4]}+1\right),$$

$$\hat{f}_{aw}(p_2^*) = \frac{(1+p_2^* MS_{e_{AB}}/MS_{ABC})^2}{\frac{1}{(a-1)(b-1)(c-1)} + \frac{(p_2^* MS_{e_{AB}}/MS_{ABC})^2}{(a-1)(b-1)(r-1)}}. \tag{10}$$

With C, using $\hat{f}_s$, the *df* for the numerator and denominator will be respectively:

$$v_1 = \frac{(MS_C + MS_{ABC})^2}{\frac{MS_C^2}{c-1} + \frac{MS_{ABC}^2}{(a-1)(b-1)(c-1)}},$$
$$v_2 = \frac{(MS_{AC} + MS_{BC})^2}{\frac{MS_{AC}^2}{(a-1)(c-1)} + \frac{MS_{ABC}^2}{(a-1)(b-1)(c-1)}}. \tag{11}$$

For $\hat{f}_{aw}$ these were the estimators for the *df* of the numerator when $MS_1 = MS_C$ and $MS_2 = MS_{ABC}$:

$$p_1 = \frac{(a-1)(b-1)(c-1)}{(a-1)(b-1)(c-1)-2}\left(\frac{2[(a-1)(b-1)(c-1)+c-3]}{(c-1)[(a-1)(b-1)(c-1)-4]}+1\right),$$

$$\hat{f}(p_1) = \frac{(1+p_1 MS_{ABC}/MS_C)^2}{\frac{1}{c-1}+\frac{(p_1 MS_{ABC}/MS_C)^2}{(a-1)(b-1)(c-1)}}; \tag{12}$$

still for the numerator, but exchanging the order of MS_1 and MS_2, we obtain:

$$p_1^* = \frac{c-1}{c-3}\left(\frac{2[(a-1)(b-1)(c-1)+c-3]}{(a-1)(b-1)(c-1)(c-5)}+1\right),$$

$$\hat{f}(p_1^*) = \frac{(1+p_1^* MS_C/MS_{ABC})^2}{\frac{1}{(a-1)(b-1)(c-1)}+\frac{(p_1^* MS_C/MS_{ABC})^2}{c-1}}. \tag{13}$$

For the denominator, taking $MS_1 = MS_{AC}$ and $MS_2 = MS_{BC}$, we get the following estimations:

$$p_2 = \frac{(b-1)(c-1)}{(b-1)(c-1)-2}\left(\frac{2\{(c-1)[(a-1)+(b-1)]-2\}}{(a-1)(c-1)[(b-1)(c-1)-4]}+1\right)$$

$$\hat{f}(p_2) = \frac{(1+p_2 MS_{BC}/MS_{AC})^2}{\frac{1}{(a-1)(c-1)}+\frac{(p_2 MS_{BC}/MS_{AC})^2}{(b-1)(c-1)}}; \tag{14}$$

once again for the denominator, but exchanging to $MS_1 = MS_{BC}$ and $MS_2 = MS_{AC}$, we get:

$$p_2^* = \frac{(a-1)(c-1)}{(a-1)(c-1)-2}\left(\frac{2\{(c-1)[(a-1)+(b-1)]-2\}}{(b-1)(c-1)[(a-1)(c-1)-4]}+1\right),$$

$$\hat{f}(p_2^*) = \frac{(1+p_2^* MS_{AC}/MS_{BC})^2}{\frac{1}{(b-1)(c-1)}+\frac{(p_2^* MS_{AC}/MS_{BC})^2}{(a-1)(c-1)}}. \tag{15}$$

3.3 F tests when only one factor has fixed effects

With respect to Table 3, the only difference for the three cases considered here (only A has fixed effects, only B has fixed effects, and only C has fixed effects) will occur in the row corresponding to the fixed effect: first, obviously, its effect will be f instead of r; second, its null hypothesis will be about the equality of all treatments inside that factor. So when A is the only factor of fixed effects, its effect is f and its null hypothesis is $A_1 = \cdots = A_a = 0$, all other fields remaining equal to Table 3; when B is the only factor with fixed effects, its effect is f and its null hypothesis is $B_1 = \cdots = B_b = 0$, all other fields remaining equal to Table 3; and when the only fixed effects are those corresponding to C, its value at effect is f and the null hypothesis will be $C_1 = \cdots = C_c = 0$, all other fields remaining equal to Table 3. This can be easily verified with the information in Subsections 2.3, 2.4 and 2.5

Since, in particular, the structure of the complex variance estimators is identical to the structure of the model with random effects, the approximate *df* for each of these three cases are exactly the same to those found in Subsection 3.2.

Table 4: F tests when only A is random

Fuente	Efecto	F	H_0
R	r	$\frac{MS_R+MS_{e_{AB}}}{MS_{e_A}+MS_{e_B}}$	$\sigma_R^2=0$
A	r	$\frac{MS_A+MS_{e_{AB}}+MS_{ABC}}{MS_{e_A}+MS_{AB}+MS_{AC}}$	$\sigma_A^2=0$
e_A	r	$\frac{MS_{e_A}}{MS_{e_{AB}}}$	$\sigma_{e_A}^2=0$
B	f	$\frac{MS_B+MS_{e_{AB}}}{MS_{e_B}+MS_{AB}}$	$B_1=B_2=\cdots=B_b=0$
e_B	r	$\frac{MS_{e_B}}{MS_{e_{AB}}}$	$\sigma_{e_B}^2=0$
AB	r	$\frac{MS_{AB}+MS_{e_t}}{MS_{e_{AB}}+MS_{ABC}}$	$\sigma_{AB}^2=0$
e_{AB}	r	$\frac{MS_{e_{AB}}}{MS_{e_t}}$	$\sigma_{e_{AB}}^2=0$
C	f	$\frac{MS_C}{MS_{AC}}$	$C_1=C_2=\cdots=C_c=0$
AC	r	$\frac{MS_{AC}}{MS_{ABC}}$	$\sigma_{AC}^2=0$
BC	f	$\frac{MS_{BC}}{MS_{ABC}}$	$(BC)_{jk}=0, \forall j,k.$
ABC	r	$\frac{MS_{ABC}}{MS_{e_t}}$	$\sigma_{ABC}^2=0$
e_t	r	$-$	

3.4 F tests when only A has random effects

When the effects of A are random, we obtain Table 4 based on the $E(MS)$'s found in Subsection 2.6.

The estimators for the *df* of A are those in (4). The estimators of the *df* by Satterthwaite for AB are those in (6); the estimators by Ames-Webster are given in equations (7) and (8) for the numerator, and (9) and (10) for the denominator. Now we procede to evaluate the *df* for the F test of B, first by means of the Satterthwaite estimator in equation (2):

$$
\begin{aligned}
v_1 &= \frac{(MS_B+MS_{e_{AB}})^2}{\frac{MS_B^2}{b-1}+\frac{MS_{e_{AB}}^2}{(a-1)(b-1)(r-1)}},\\
v_2 &= \frac{(MS_{e_B}+MS_{AB})^2}{\frac{MS_{e_B}^2}{(b-1)(r-1)}+\frac{MS_{AB}^2}{(a-1)(b-1)}}.
\end{aligned} \tag{16}
$$

Still with B, the first Ames-Webster estimator for the *df* of the numerator of the F, taking $MS_1 = MS_B$ and $MS_2 = MS_{e_{AB}}$ will be:

$$
p_1 = \frac{(a-1)(b-1)(r-1)}{(a-1)(b-1)(r-1)-2}\left(\frac{2\{(b-1)[(a-1)(r-1)+1]-2\}}{(b-1)[(a-1)(b-1)(r-1)-4]}+1\right),
$$

$$
\hat{f}(p_1) = \frac{(1+p_1 MS_{e_{AB}}/MS_B)^2}{\frac{1}{b-1}+\frac{(p_1 MS_{e_{AB}}/MS_B)^2}{(a-1)(b-1)(r-1)}};
$$

and exchanging the order to $MS_1 = MS_{e_{AB}}$ and $MS_2 = MS_B$, we obtain:

$$
p_1^* = \frac{b-1}{b-3}\left(\frac{2\{(b-1)[(a-1)(r-1)+1]-2\}}{(a-1)(b-1)(r-1)(b-5)}+1\right),
$$

$$
\hat{f}(p_1^*) = \frac{(1+p_1^* MS_B/MS_{e_{AB}})^2}{\frac{1}{(a-1)(b-1)(r-1)}+\frac{(p_1^* MS_B/MS_{e_{AB}})^2}{b-1}}.
$$

For the denominator of the the F test of B, taking $MS_1 = MS_{e_B}$ y $MS_2 = MS_{AB}$:

$$p_2 = \frac{(a-1)(b-1)}{(a-1)(b-1)-2}\left(\frac{2[(b-1)(a+r-2)-2]}{(b-1)(r-1)[(a-1)(b-1)-4]}+1\right),$$

$$\hat{f}(p_2) = \frac{(1+p_2 MS_{AB}/MS_{e_B})^2}{\frac{1}{(b-1)(r-1)} + \frac{(p_2 MS_{AB}/MS_{e_B})^2}{(a-1)(b-1)}};$$

and exchanging the order of MS_1 and MS_2:

$$p_2^* = \frac{(b-1)(r-1)}{(b-1)(r-1)-2}\left(\frac{2[(b-1)(a+r-2)-2]}{(a-1)(b-1)[(b-1)(r-1)-4]}+1\right),$$

$$\hat{f}(p_2^*) = \frac{(1+p_2^* MS_{e_B}/MS_{AB})^2}{\frac{1}{(a-1)(b-1)} + \frac{(p_2^* MS_{e_B}/MS_{AB})^2}{(b-1)(r-1)}}.$$

3.5 F tests when only B has random effects

When only A and C have fixed effects, based on Subsection (2.7), we get Table 5.

Table 5: F tests when only B is random

Source	Effect	F	H_0
R	r	$\frac{MS_R+MS_{e_{AB}}}{MS_{e_A}+MS_{e_B}}$	$\sigma_R^2 = 0$
A	f	$\frac{MS_A+MS_{e_{AB}}}{MS_{e_A}+MS_{AB}}$	$A_1 = A_2 = \cdots = A_a = 0$
e_A	r	$\frac{MS_{e_A}}{MS_{e_{AB}}}$	$\sigma_{e_A}^2 = 0$
B	r	$\frac{MS_B+MS_{e_{AB}}+MS_{ABC}}{MS_{e_B}+MS_{AB}+MS_{BC}}$	$\sigma_B^2 = 0$
e_B	r	$\frac{MS_{e_B}}{MS_{e_{AB}}}$	$\sigma_{e_B}^2 = 0$
AB	r	$\frac{MS_{AB}+MS_{e_t}}{MS_{e_{AB}}+MS_{ABC}}$	$\sigma_{AB}^2 = 0$
e_{AB}	r	$\frac{MS_{e_{AB}}}{MS_{e_t}}$	$\sigma_{e_{AB}}^2 = 0$
C	f	$\frac{MS_C}{MS_{BC}}$	$C_1 = C_2 = \cdots = C_c = 0$
AC	f	$\frac{MS_{AC}}{MS_{ABC}}$	$(AC)_{ik} = 0, \forall i,k.$
BC	r	$\frac{MS_{BC}}{MS_{ABC}}$	$\sigma_{BC}^2 = 0$
ABC	r	$\frac{MS_{ABC}}{MS_{e_t}}$	$\sigma_{ABC}^2 = 0$
e_t	r	–	

The approximated *df* for the F test of B were found using the Satterthwaite estimator (5). The approximation of the *df* for AB using Satterthwaite is given by (6); using Ames-Webster, the estimator for the *df* of AB are given in equations (7) and (8) for the numerator, and (9) and (10) for the denominator. Now we procede to evaluate the *df* for the F test of A, first by means of the Satterthwaite estimator in equation (2):

$$v_1 = \frac{(MS_A + MS_{e_{AB}})^2}{\frac{MS_A^2}{a-1} + \frac{MS_{e_{AB}}^2}{(a-1)(b-1)(r-1)}},$$

$$v_2 = \frac{(MS_{e_A} + MS_{AB})^2}{\frac{MS_{e_A}^2}{(r-1)(a-1)} + \frac{MS_{AB}^2}{(a-1)(b-1)}}.$$

The Ames-Webster estimator for the numerator is the following when $MS_1 = MS_A$ and $MS_2 = MS_{e_{AB}}$:

$$p_1 = \frac{(a-1)(b-1)(r-1)}{(a-1)(b-1)(r-1)-2}\left(\frac{2\{(a-1)[(b-1)(r-1)+1]-2\}}{(a-1)[(a-1)(b-1)(r-1)-4]}+1\right),$$

$$\hat{f}(p_1) = \frac{(1+p_1 MS_{e_{AB}}/MS_A)^2}{\frac{1}{a-1}+\frac{(p_1 MS_{e_{AB}}/MS_A)^2}{(a-1)(b-1)(r-1)}};$$

still with the numerator but taking $MS_1 = MS_{e_{AB}}$ and $MS_2 = MS_A$, we get:

$$p_1^* = \frac{a-1}{a-3}\left(\frac{2\{(a-1)[(b-1)(r-1)+1]-2\}}{(a-1)(b-1)(r-1)(a-5)}+1\right),$$

$$\hat{f}(p_1^*) = \frac{(1+p_1^* MS_A/MS_{e_{AB}})^2}{\frac{1}{(a-1)(b-1)(r-1)}+\frac{(p_1^* MS_A/MS_{e_{AB}})^2}{a-1}}.$$

For the denominator, doing $MS_1 = MS_{e_A}$ and $MS_2 = MS_{AB}$, we get:

$$p_2 = \frac{(a-1)(b-1)}{(a-1)(b-1)-2}\left(\frac{2[(a-1)(b+r-2)-2]}{(a-1)(r-1)[(a-1)(b-1)-4]}+1\right),$$

$$\hat{f}(p_2) = \frac{(1+p_2 MS_{AB}/MS_{e_A})^2}{\frac{1}{(a-1)(r-1)}+\frac{(p_2 MS_{AB}/MS_{e_A})^2}{(a-1)(b-1)}};$$

finally, exchanging the order of MS_1 and MS_2, we obtain:

$$p_2^* = \frac{(a-1)(r-1)}{(a-1)(r-1)-2}\left(\frac{2[(a-1)(b+r-2)-2]}{(a-1)(b-1)[(a-1)(r-1)-4]}+1\right),$$

$$\hat{f}(p_2^*) = \frac{(1+p_2^* MS_{e_A}/MS_{AB})^2}{\frac{1}{(a-1)(b-1)}+\frac{(p_2^* MS_{e_A}/MS_{AB})^2}{(a-1)(r-1)}}.$$

3.6 F tests when only C has random effects

Table 6 was constructed using the $E(MS)$'s in Subsection 2.8. Note that the approximate degrees of freedom for AB were described in equations (6) by means of Satterthwaite; also for AB, the approximations of its degrees of freedom using Ames-Webster were given in equations (7) and (8) for the numerator , and (9) and 10 for the denominator.

For C, its approximate *df* using Satterthwaite were found in (11). And the Ames-Webster estimators of the *df* of C are given by (12) and (13) for the numerator, and by (14) and (15) for the denominator.

We procede to evaluate the approximate *df* for the F test of A, first by means of the Satterthwaite estimator in equation (2):

Table 6: F tests when only C is random

Source	Effect	F	H_0
R	r	$\frac{MS_R+MS_{e_{AB}}}{MS_{e_A}+MS_{e_B}}$	$\sigma^2_R=0$
A	f	$\frac{MS_A+MS_{e_t}}{MS_{e_A}+MS_{AC}}$	$A_1=A_2=\cdots=A_a=0$
e_A	r	$\frac{MS_{e_A}}{MS_{e_{AB}}}$	$\sigma^2_{e_A}=0$
B	f	$\frac{MS_B+MS_{e_t}}{MS_{e_B}+MS_{BC}}$	$B_1=B_2=\cdots=B_b=0$
e_B	r	$\frac{MS_{e_B}}{MS_{e_{AB}}}$	$\sigma^2_{e_B}=0$
AB	f	$\frac{MS_{AB}+MS_{e_t}}{MS_{e_{AB}}+MS_{ABC}}$	$(AB)_{ij}=0, \forall i,j.$
e_{AB}	r	$\frac{MS_{e_{AB}}}{MS_{e_t}}$	$\sigma^2_{e_{AB}}=0$
C	r	$\frac{MS_C+MS_{ABC}}{MS_{AC}+MS_{BC}}$	$\sigma^2_C=0$
AC	r	$\frac{MS_{AC}}{MS_{ABC}}$	$\sigma^2_{AC}=0$
BC	r	$\frac{MS_{BC}}{MS_{ABC}}$	$\sigma^2_{BC}=0$
ABC	r	$\frac{MS_{ABC}}{MS_{e_t}}$	$\sigma^2_{ABC}=0$
e_t	r	$-$	

$$v_1=\frac{(MS_A+MS_{e_t})^2}{\frac{MS_A^2}{a-1}+\frac{MS_{e_t}^2}{ab(c-1)(r-1)}},$$

$$v_2=\frac{(MS_{e_A}+MS_{AC})^2}{\frac{MS_{e_A}^2}{(r-1)(a-1)}+\frac{MS_{AC}^2}{(a-1)(c-1)}}.$$

The Ames-Webster estimator for the numerator is the following when $MS_1=MS_A$ and $MS_2=MS_{e_t}$:

$$p_1=\frac{ab(c-1)(r-1)}{ab(c-1)(r-1)-2}\left(\frac{2[ab(c-1)(r-1)+a-3]}{(a-1)[ab(c-1)(r-1)-4]}+1\right),$$

$$\hat{f}(p_1)=\frac{(1+p_1MS_{e_t}/MS_A)^2}{\frac{1}{a-1}+\frac{(p_1MS_{e_t}/MS_A)^2}{ab(c-1)(r-1)}};$$

still with the numerator but taking $MS_1=MS_{e_t}$ and $MS_2=MS_A$, we get:

$$p_1^*=\frac{a-1}{a-3}\left(\frac{2[ab(c-1)(r-1)+a-3]}{ab(c-1)(r-1)(a-5)}+1\right),$$

$$\hat{f}(p_1^*)=\frac{(1+p_1^*MS_A/MS_{e_t})^2}{\frac{1}{ab(c-1)(r-1)}+\frac{(p_1^*MS_A/MS_{e_t})^2}{a-1}}.$$

For the denominator, doing $MS_1=MS_{e_A}$ and $MS_2=MS_{AC}$, we get:

$$p_2=\frac{(a-1)(c-1)}{(a-1)(c-1)-2}\left(\frac{2[(a-1)(r+c-2)-2]}{(a-1)(r-1)[(a-1)(c-1)-4]}+1\right),$$

$$\hat{f}(p_2) = \frac{(1 + p_2 MS_{AC}/MS_{e_A})^2}{\frac{1}{(a-1)(r-1)} + \frac{(p_2 MS_{AC}/MS_{e_A})^2}{(a-1)(c-1)}};$$

finally, exchanging the order of MS_1 and MS_2, we obtain:

$$p_2^* = \frac{(a-1)(r-1)}{(a-1)(r-1)-2}\left(\frac{2[(a-1)(r+c-2)-2]}{(a-1)(c-1)[(a-1)(r-1)-4]} + 1\right),$$

$$\hat{f}(p_2^*) = \frac{(1 + p_2^* MS_{e_A}/MS_{AC})^2}{\frac{1}{(a-1)(c-1)} + \frac{(p_2^* MS_{e_A}/MS_{AC})^2}{(a-1)(r-1)}}.$$

Now, we evaluate the approximate *df* for the F test of B, first by means of the Satterthwaite estimator in equation (2):

$$v_1 = \frac{(MS_B + MS_{e_t})^2}{\frac{MS_B^2}{b-1} + \frac{MS_{e_t}^2}{ab(c-1)(r-1)}},$$

$$v_2 = \frac{(MS_{e_B} + MS_{BC})^2}{\frac{MS_{e_B}^2}{(r-1)(b-1)} + \frac{MS_{BC}^2}{(b-1)(c-1)}}.$$

The Ames-Webster estimator for the numerator is the following when $MS_1 = MS_B$ and $MS_2 = MS_{e_t}$:

$$p_1 = \frac{ab(c-1)(r-1)}{ab(c-1)(r-1)-2}\left(\frac{2[ab(c-1)(r-1)+b-3]}{(b-1)[ab(c-1)(r-1)-4]} + 1\right),$$

$$\hat{f}(p_1) = \frac{(1 + p_1 MS_{e_t}/MS_B)^2}{\frac{1}{b-1} + \frac{(p_1 MS_{e_t}/MS_B)^2}{ab(c-1)(r-1)}};$$

still with the numerator but taking $MS_1 = MS_{e_t}$ and $MS_2 = MS_B$, we get:

$$p_1^* = \frac{b-1}{b-3}\left(\frac{2[ab(c-1)(r-1)+b-3]}{ab(c-1)(r-1)(b-5)} + 1\right),$$

$$\hat{f}(p_1^*) = \frac{(1 + p_1^* MS_B/MS_{e_t})^2}{\frac{1}{ab(c-1)(r-1)} + \frac{(p_1^* MS_B/MS_{e_t})^2}{b-1}}.$$

For the denominator, doing $MS_1 = MS_{e_B}$ and $MS_2 = MS_{BC}$, we get:

$$p_2 = \frac{(b-1)(c-1)}{(b-1)(c-1)-2}\left(\frac{2[(b-1)(r+c-2)-2]}{(b-1)(r-1)[(b-1)(c-1)-4]} + 1\right),$$

$$\hat{f}(p_2) = \frac{(1 + p_2 MS_{BC}/MS_{e_B})^2}{\frac{1}{(b-1)(r-1)} + \frac{(p_2 MS_{BC}/MS_{e_B})^2}{(b-1)(c-1)}};$$

finally, exchanging the order of MS_1 and MS_2, we obtain:

$$p_2^* = \frac{(b-1)(r-1)}{(b-1)(r-1)-2}\left(\frac{2[(b-1)(r+c-2)-2]}{(b-1)(c-1)[(b-1)(r-1)-4]} + 1\right),$$

$$\hat{f}(p_2^*) = \frac{(1 + p_2^* MS_{e_B}/MS_{BC})^2}{\frac{1}{(b-1)(c-1)} + \frac{(p_2^* MS_{e_B}/MS_{BC})^2}{(b-1)(r-1)}}.$$

4 Application

In Zimmermann (2004), a real example was considered when all the effects are fixed. The data in Tables 7 and 8 show the weight of 100 beans obtained by Luis Fernando Stone and Regis Vilela Bagatini on an experiment in 1998. It is a complete block design with two replicates on which each horizontal strip corresponds to the water layer irrigated, the vertical strips are soil tillage systems and the subplots are Nitrogen doses. The experiment was done at the Capivara farm in Embrapa Rice and Bean.

Table 7: Block 1

Water	Soil 1			Soil 2			Soil 3		
	Nit 1	Nit 2	Nit 3	Nit 1	Nit 2	Nit 3	Nit 1	Nit 2	Nit 3
Water 1	26.33	27.85	27.13	25.10	27.67	24.93	25.00	28.03	29.65
Water 2	24.04	25.22	28.32	25.19	27.77	27.28	25.89	24.27	25.83
Water 3	25.85	25.70	26.97	25.63	27.11	25.62	26.16	24.86	25.51
Water 4	23.20	20.32	23.94	29.28	26.03	28.60	26.23	25.49	24.65

Table 8: Block 2

Water	Soil 1			Soil 2			Soil 3		
	Nit 1	Nit 2	Nit 3	Nit 1	Nit 2	Nit 3	Nit 1	Nit 2	Nit 3
Water 1	25.87	28.64	29.31	27.80	27.25	25.56	28.53	26.38	32.45
Water 2	27.16	26.49	25.99	24.63	26.91	28.47	26.68	27.64	24.80
Water 3	27.11	24.44	28.06	25.77	27.46	26.20	26.83	27.55	27.19
Water 4	23.00	23.43	23.42	28.71	26.45	26.25	26.64	26.82	26.88

The water layers (the vertical strips, A) are averaged irrigation levels as follows: 366.1 mm for the first horizontal strip, 335.1 mm for the second one, 315.7 mm for the third one, and 293.7 mm for the last one. There are three ways to prepare the soil (the vertical strips B): heavy harrowing for the first vertical strip, moldboard plowing for the second one, and notillage on the last one. The Nitrogen subdoses (C) inside the subplots are, respectively for each subplot, 0, 20 and 40 kg ha^{-1}. We present a SAS program for the situation just considered:

```
data a;
input bloque trata tratb tratc x1;
cards;
.....
.....
;
proc anova;class bloque trata tratb tratc;
model x1 = bloque trata bloque*trata tratb bloque*trab
trata*tratb bloque*trata*tratb
tratc trata*tratc tratb*tratc trata*trab*tratc;
test h=trata e=bloque*trata;
test h=tratb e=bloque*trab;
test h=trata*tratb e=bloque*trata*tratb;
run;
quit;
```

The MS and the df needed to construct the F tests are shown on Table 9. These results show that there are significant effects on the water layer, its interaction with the soil, and the interaction of the three factors.

Table 9: Example

Source	df	MS	F (Pr $>F$)
R	1	9.4758	
A	3	10.9903	26.04 (0.0119)
e_A	3	0.4220	
B	2	7.3937	2.91 (0.2556)
e_B	2	2.5387	
AB	6	11.2718	35.89 (0.0002)
e_{AB}	6	0.3141	
C	2	3.1476	2.11 (0.1432)
AC	6	2.3759	1.59 (0.1926)
BC	4	1.8678	1.25 (0.3161)
ABC	12	3.2911	2.21 (0.0479)
e_t	24	1.4921	

References

AMES M., WEBSTER J.T. (1991) On Estimating Approximate Degrees of Freedom. *The American Statistician.* **45**, pp 45–50.

COCHRAN W.G., COX G. (1992) Experimental Designs, second edition. *Wiley*

DÍAZ-PACHÓN D.A., (2004) Hipótesis lineales sobre medias para experimentos de franjas en parcelas divididas, Universidad Nacional de Colombia, Thesis of Statistics.

GOMEZ A., GOMEZ K. (1984) Statistical Procedures for Agricultural Research, second edition, *Wiley & Sons*, pp 154–167.

KUEHL R.O. (1999) Design of Experiments: Statistical Principles of Research Design and Analysis, second edition, *Duxbury.*

McINTONSH, M.S. (1983) Analysis of Combined Experiments. *Agronomy Journal.* **75**, pp 153–155.

MOSER B.K., SAWYER J.K. (1998) Algorithms for Sums of Squares and Covariance Matrices Using Kronecker Products. *The American Statistician.* **52**, pp. 54–57.

MONTGOMERY D.C. (2012) Design and Analysis of Experiments, eight edition, *Wiley & Sons.*

SAAVEDRA A. (2000) Contrastes lineales de medias en experimentos combinados – Caso Parcelas Divididas, Universidad Nacional de Colombia, Thesis of Statistics.

SATTERTHWAITE F.E. (1946) An Approximate Distribution of Estimates of Variance Components. *Bull. Biometrics.* **2**, pp 110–114.

SEARLE S.R., CASELLA G., McCULLOCH, C. (1992) Variance components, *Wiley & Sons.*

ZIMMERMANN, F.J.P. (2004) Estadística Aplicada a la Investigación, *Universidad de la Sabana.*

ZIMMERMANN F.J.P. (2014) Estatística Aplicada ‡ Pesquisa Agrícola. Seconda ed., revista e ampliada, *Embrapa*.