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Is Cosmological Tuning Fine or Coarse?

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Abstract. The fine-tuning of the universe for life, the idea that the constants of nature (or ratios between them) must belong to very small intervals in order for life to exist, has been debated by scientists for several decades. Several criticisms have emerged concerning probabilistic measurement of life-permitting intervals. Herein, a Bayesian statistical approach is used to assign an upper bound for the probability of tuning, which is invariant with respect to change of physical units, and under certain assumptions it is small whenever the life-permitting interval is small on a relative scale. The computation of the upper bound of the tuning probability is achieved by first assuming that the prior is chosen by the principle of maximum entropy (MaxEnt). The MaxEnt assumption is "maximally noncommittal with regard to missing information." This approach is sufficiently general to be applied to constants of current cosmological models, or for other constants possibly under different models. Application of the MaxEnt model reveals, for example, that the ratio of the universal gravitational constant to the square of the Hubble constant is finely tuned in some cases, whereas the amplitude of primordial fluctuations is not.

Keywords: Constants of nature, fine-tuning, maximum entropy, Bayesian statistics.

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Contents

1 General Introduction

The fine-tuning of the universe for life argues that the constants in the laws of nature or the ratios thereof, and/or the boundary conditions, both in particle physics and in the standard cosmological model, belong to minuscule life-permitting intervals such that outside them life could not exist. The modern version of fine-tuning was introduced by Carter [1]. Since inception, fine-tuning remains a hot topic among scientists and popularizers [2–5]. Perhaps fine-tuning's biggest claim to fame came with the book *The Anthropic Cosmological Principle*, by Barrow and Tipler [6]. Subsequently the argument has been scrutinized in physics and cosmology (see, for instance, [7–9] and all the references therein).

Even though the definition of fine-tuning appears to be simple, there are variations in the particulars. On one hand, researchers have differing opinions inasmuch as the constants of nature that must be considered. Just to mention one instance out of many, Rees [10] and Adams [7] consider the gravitational constant, but Tegmark et al. [9] and Barnes [11] ignore it, limiting themselves to the constants in the Standard Models. On the other hand, when talking about life, consensus is illusive largely because there is not even a consistent definition of what life is. As an illustration, Adams extends his argument to any conjectured form of life, not necessarily carbon-based [7]; Sandora focuses on complex intelligent life [12–15]; and philosopher Robin Collins talks about embodied moral agents [16].

However, in spite of fine-tuning ado and the passions elicited in critics and defenders, the degree of tuning, either fine or coarse, remains unsettled. To emphasize, the claim is not that the tuning is fine —or coarse. The claim is that, up to this point, the degree of tuning has yet to be determined. In order to explain this assertion, let's examine in more detail how the tuning should be measured. Assuming the constants to be considered are agreed upon, the process can be summarized in two steps.

1. The bounds for the life-permitting intervals of constants or of their ratios must be identified.

2. Probabilities over life-permitting intervals must be calculated.

Both steps require elaboration.

For the first step, physicists have proposed useful boundaries of the life-permitting intervals for many of the constants or the ratios of constants [8]. We must add, however, that in the absence of a theory of everything the task is far from complete. Many of these numbers could change in future research in at least three ways: (i) by changing the limits of the life-permitting intervals, (ii) by removing constants because they have lost relevance, or (iii) by identification of constants hitherto before not considered.

We focus on the second step of finding the degree of tuning. More specifically, in order to determine the probability of having a constant of nature or the ratio between two constants of nature within a life-permitting interval, a Bayesian approach is used. Either the prior distribution of the constant is chosen by the principle of maximum entropy (MaxEnt), or two constants forming a ratio are chosen to have two separate MaxEnt distributions. Such an approach allows finding an upper bound, $P_{\max}$, for the probability of tuning. This upper bound is dimensionless, i.e. invariant with respect to changing physical units. It is very small for some (but not all) examples, and we refer to tuning as being fine or coarse depending on whether $P_{\max}$ is small or not. Our proposed approach is sufficiently general to be applied to the constants in the current (standard) models or to different constants if they change by any of the three aforementioned circumstances.

2 Introduction of Probabilities to Fine-Tuning

We do not consider the first step, finding the bounds of life-permitting intervals, relying instead on what has already been identified. Our analysis is restricted to the second step — how to calculate the probabilities of tuning given that the life-permitting intervals have been determined. The problem appears to still be open. For instance, Adams halts his presentation at the point of finding the life-permitting intervals, explaining that not much can be said of the probability distributions that must govern the behavior of the constants of nature [7, p. 6]:

> A full assessment of fine-tuning requires knowledge of these fundamental probability distributions, one for each [constant of nature] of interest (although they are not necessarily independent). These probability distributions, however, are not currently assessable.

In the absence of these probabilities, not much can be said about the degree of tuning. As surprising as a small interval might intuitively seem, the interval size matters less than the probability over the interval. For instance, for a standard normal distribution, the intervals $(-\infty, 0)$ and $(-0.6745, 0.6745)$ each have probability 0.5 even though the first has infinite length and the second has length 1.349. In contrast, under this distribution, the interval $\left(-\infty, -10^9\right)$, has infinite length but almost zero probability.

In addition, there is the normalization problem [17]. Normalization imposes limitations to Bernoulli's *Principle of Insufficient Reason* (PrOIR), otherwise known as the *Principle of Indifference* [18–20]. As originally conceived, the PrOIR states that in the absence of any prior knowledge events must be uniformly distributed. The unspoken assumption is that the space must be finite. Normalization posits a general warning against using the PrOIR beyond finite spaces. Placing a uniform distribution over the whole real line, for example, is untenable. McGrew et al. [17] raise the normalization objection in the context of fine-tuning. Since the space where the constants of nature or their ratios could take values has infinite length, attempts to find the probabilities using a uniform distribution cannot be successful. This criticism against the PrOIR in the fine-tuning literature is legitimate.

Yet some persist, for instance, borrowing an idea from quantum field theory in his analysis of the initial entropy of the universe, Roger Penrose reduces an infinite-dimensional phase space $\mathcal{P}_{\mathcal{U}}$ to a new finite-dimensional space in which each dimension has finite size.[1]

Tegmark et al. [9] have addressed the second step too. They propose to calculate the probability that a constant of nature x belongs to its life-permitting interval by decomposing the density function of x as a product of a prior density and a selection probability ($f(x) \propto f_{prior}(x) f_{sel}(x)$). Here the prior is a theoretically predicted distribution at some random point during inflation and the selection distribution obtains the probability of observing that point. This second term is related to the weak anthropic principle. Observers in a universe are bound to measure constants of nature for which a habitable universe is possible with a nonzero f_{sel}. As an example, Tegmark et al. consider the probability distribution of the vacuum energy density assuming uniformity. However, other distributions are possible [23].

Recently, Barnes has fleshed out these probabilities using a Bayesian approach [8, 24–26]. Barnes, as Collins [16], does what McGrew et al. [17] criticize by assuming finite sample spaces, therefore using uniform distributions to calculate probabilities. Barnes has been the first to justify his use of a continuous uniform (or what physicists call a "flat" distribution which is the MaxEnt distribution in a space of finite size absent all other knowledge) following Edwin Jaynes's recommendations [27].

Throughout this article a Bayesian approach is applied [28]. Using informational maximum entropy, a concept also due to Jaynes [29, 30], we directly assume an infinite space using the domain for a class of distributions over the relevant space. Maximum entropy generalizes the PrOIR. Maximum entropy over an infinite domain is no longer uniform but is still applicable under the appropriate MaxEnt distribution.

A clarification should be made that entropy as treated here refers to information entropy, not to thermodynamics entropy. In thermodynamics, one uses entropy in a slightly different

[1] With respect to converting the space to a finite-dimensional one Penrose writes "In fact $\mathcal{P}_{\mathcal{U}}$ will be infinite dimensional... This causes some technical problems for the definition of entropy, since each required phase space region $\mathcal{V}$ will have infinite volume. It is usual to deal with this problem by borrowing ideas from quantum (field) theory, which enables a finite answer to be obtained for the phase-space volumes which refer to systems that are appropriately bounded in energy and spatial dimension... Although there is no fully satisfactory way of dealing with these issues in the case of gravity—owing to a lack of a satisfactory theory of quantum gravity—I am going to regard these as technicalities that do not affect the general discussion of the issues raised by the second law." [21, pp. 700-701]. See also the more formal result in [22]. And regarding making each dimension to have finite size he adds: "We shall use the phase space $\mathcal{P}_{\mathcal{U}}$ appropriate to the entire universe, so the evolution of the universe as a whole is described by the point x moving along a curve ξ in $\mathcal{P}_{\mathcal{U}}$. The curve ξ is parametrized by the time coordinate t, and we can expect that, from the second law, ξ enters immensely larger and larger coarse-graining boxes as t increases. We suppose that some 'reasonable' coarse graining has been applied to $\mathcal{P}_{\mathcal{U}}$, but if we wish to obtain finite values for the entropies that x encounters, we would want the volumes of these boxes to be finite." [21, pp. 701-702].

way than here: Randomness refers to the *degree of disorder of a physical system*, and the second law of thermodynamics states that the entropy of a closed system, without external influences, will increase towards an equilibrium state of maximal disorder. In this paper randomness refers to *epistemic uncertainty* regarding the value of a constant of nature, or the ratio between two such constants. In particular, entropy in information is an inherent property of the distribution function that describes this epistemic uncertainty, and it corresponds to our degree of ignorance. This perspective allows to extend entropy even to distributions of random variables whose domain is not of finite (counting or Lebesgue) measure.

Then, with appropriate restrictions applied to the moments of distributions in non-compact spaces, it is possible to find even in these settings MaxEnt distributions. In particular, for our case of interest, maximum entropy for the distribution of the constants of nature or ratios then refers to a maximum degree of ignorance about their values. As Jaynes saw it, entropy in statistical mechanics is but an application of the more general information concept [29].

Even though our main focus in this article is epistemic uncertainty, it should be added that MaxEnt distributions for physical systems are observed in nature. As we saw, the second law of thermodynamics famously states that the entropy of a gas in a closed room reaches maximum entropy in pressure in accordance to a uniform PrOIR. What if, on the other hand, there is only a single boundary? The barometric pressure measured from the surface of the earth to space follows the MaxEnt distribution of $\mathrm{Exp}(1/\mu)$. An example of MaxEnt that is unbounded for negative and positive values is the Maxwell-Boltzmann distribution describing the velocity of ideal gas particles. The projection of the velocity vector along each direction is unbounded with a $\mathcal{N}(0, \sigma^2)$ distribution, which is MaxEnt for domains bounded on neither side (Table 1 below and [31].)

3 Fine-Tuning and MaxEnt

Fine-tuning asserts that the constants of nature and the boundary conditions of the universe must live in narrow intervals of low probability in order to make life possible and that if such constants would not have had the actual values they possess, life would have never been possible. Therefore, fine-tuning arguments assume that the value of a constant of nature, or the ratio between two constants, let's call it $x = X_{\text{obs}}$, is an observation of a random variable X. Then, for X, a probability of the life-permitting interval is calculated.

As seen in the previous section, the interval width of the life-permitting interval is less important than the corresponding probability. Denote the life-permitting interval of X as ℓ_X, and its length as $|\ell_X|$. In order to properly assess this probability, three steps (I-III) need to be followed:

I) Determine the *right* sample space Ω for X. The incorrect determination of Ω is what McGrew et al. [17] criticized, and ad-hoc attempts to force a finite sample space from an infinite one do not, as noted by McGrew and McGrew [32], seem convincing. Proceeding is not possible without first providing the right sample space.

II) Find a probability distribution F of X. F must be such that it best represents the current knowledge of the behavior of a constant of nature or of the ratio between two constants in the most noncommittal way. In this step we apply, instead of the uniform assumption of the PrOIR, either the MaxEnt principle to X itself or the MaxEnt principle applied to each of the two constants whose ratio is X. Jaynes [29, p. 623] summarizes the advantages of MaxEnt over the PrOIR in this way:

> The principle of [MaxEnt] may be regarded as an extension of the [PrOIR] (to which it reduces in case no information is given except enumeration of possibilities x_i), with the following essential difference. The [MaxEnt] distribution may be asserted for the positive reason that it is uniquely determined as the one which is *maximally noncommittal with regard to missing information*, instead of the negative one that there was no reason to think otherwise. Thus the concept of entropy supplies the missing criterion of choice which Laplace needed to remove the apparent arbitrariness of the [PrOIR], and in addition shows precisely how this principle is to be modified in case there are reasons for "thinking otherwise." [Emphasis added.]

The MaxEnt principle can be applied directly to Ω for the case of consideration of a single constant. Another option is to assume that in the ratio of two physical constants, each constant is MaxEnt. MaxEnt can still find a reference distribution for unbounded domains [31, 33, 34], whereas the PrOIR cannot. Table 1 below illustrates the best-known cases of MaxEnt distributions that can be applied to constants or ratios of constants living in a unidimensional space. (A more comprehensive table can be found in [35].)

To introduce the MaxEnt Bayesian approach, first consider a problematic application to fine-tuning. Let x be an observation of a random variable X that belongs to Ω. Since the sample size is 1, one may first assume x was chosen according to F, and, as such, $x = X_{\text{obs}}$ would become the average of the sample. Being a sample of size one, this value serves as a sufficient statistic for the expected value $\mu = E(X)$ of X (see, e.g., [36, Ch. 6]). In usual statistical notation, $\hat{\mu} = x$. From the first step the space is known and, from the second step, an estimated value for the mean can be obtained.

However, this approach has a weakness due to the weak anthropic principle. Since we live in a habitable universe, x is not an unbiased observation of a random variable X with distribution F. It is rather an observation of a random variable with a truncated distribution

$$F_{\text{trunc}} = F | X \in \ell_X.$$

Then $\hat{\mu} = x$ is not an unbiased estimate of $\mu = E_F(X)$ but rather an unbiased estimate of $\mu^* = E_{F_{\text{trunc}}}(X)$. In particular, if ℓ_X is narrow, so that a uniform distribution approximates F_{trunc}, then μ^* approximately equals the mid point of ℓ_X. On the other hand, F_{trunc} may differ substantially from a uniform distribution for constants of nature with a wider ℓ_X. In such a case x is not an unbiased estimate of the mid point of ℓ_X.

These considerations lead to the proper approach: although x is not an observation of $X \sim F$, we can assume that X belongs to a class of MaxEnt distributions $F = F(\,\cdot\,;\theta)$ taken from Table 1, with θ a finite-dimensional unknown parameter that due to the weak anthropic principle cannot be estimated easily. In Bayesian statistics such a parameter on the prior distribution F is referred to as a hyperparameter [28]. Alternatively, when $X = G/D$ is the ratio of two constants of nature G and D, we can also reasonably assume that G and D are independent with distributions $F_G = F_G(\,\cdot\,;\theta_G)$ and $F_D = F_D(\,\cdot\,;\theta_D)$ chosen as MaxEnt distributions from Table 1. We need then to find the distribution $F(\,\cdot\,;\theta)$ of X, where F and the hyperparameter θ are functions of F_G, θ_G, F_D, and θ_D. In both cases, whether a MaxEnt distribution is used for X itself, or for G and D, the MaxEnt principle helps to reduce the class of possible F from an infinite class of distributions to a finite-dimensional class of distributions.

Space	Restrictions/Knowledge	MaxEnt distribution
Finite	None	Equiprobability
Finite interval $[a, b]$	None	$\mathcal{U}(a, b)$
Finite interval $[a, b]$	$\mathbf{E}X = \mu_T$; $\mathbf{E}(X - \mu_T)^2 = \sigma_T^2$	Truncated normal
$\mathbb{N}$	$\mathbf{E}X = \mu$	Geom$(1/\mu)$
$\mathbb{R}^+$	$\mathbf{E}X = \mu$	Exp$(1/\mu)$
$\mathbb{R}$	$\mathbf{E}X = \mu$; $\mathbf{E}(X - \mu)^2 = \sigma^2$	$\mathcal{N}(\mu, \sigma^2)$

Table 1. Maximum entropy distributions over some relevant spaces under different restrictions.

III) Calculate the maximum probability of the life-permitting interval ℓ_X under the class of distributions $\{F(\cdot;\theta);\ \theta \in \Theta\}$. Following [37], let A be the event "We observe a universe that exists and permits life," and let's regard x now as a parameter of the model. The tuning probability of the event A is then

$$P(A;\theta) = \int_{\Omega} P(A|x)dF(x;\theta), \tag{3.1}$$

where $dF(x;\theta) = f(x;\theta)dx$, $f(x;\theta)$ is the prior density, and $P(A|x)$ the likelihood. If $A = \ell_X$, then $P(A|x) = 1$ if $x \in A$, and $P(A|x) = 0$ if $x \notin A$. Then (3.1) reduces to the tuning probability

$$P(A;\theta) = F(\ell_X;\theta) \tag{3.2}$$

considered here. Notice in particular that we might know of ℓ_X through the experiments/research that led to ℓ_X being determined (which includes the fact that x belongs to ℓ_X). However, in any case, the event A is well defined whether we, as observers, know of ℓ_X or not.

In order to finalize the third step and determine the degree of tuning, we also need to maximize (3.2) with respect to the hyperparameter θ, when θ varies over a finite-dimensional space Θ. That is, our final degree of tuning equals

$$P_{\max} = \max_{\theta\in\Theta} F(\ell_X;\theta). \tag{3.3}$$

Notice in particular that the degree of tuning (3.3) can be calculated without violating the weak anthropic principle. This is so, since we did not estimate the hyperparameter θ from the single observation x, but rather maximized the tuning probability (3.2) with respect to θ.

Suppose the life permitting interval

$$\ell_X = x[1-\varepsilon, 1+\varepsilon] \tag{3.4}$$

is centered around x, with a relative half size ε. In the next section, it will be shown that whenever ℓ_X is small, the upper bound (3) of the tuning probability is proportional to ε, i.e.

$$P_{\max}(\varepsilon) = C\varepsilon, \tag{3.5}$$

for a constant of proportionality C that depends on the size of the family of prior distributions $\{F(\cdot,\theta);\ \theta \in \Theta\}$. Since ε is a dimensionless constant, it follows that the upper bound of the tuning probability is dimensionless as well.

4 Examples

In this section the methodology of Section 3 is applied to some specific cases using values of the width of life-permitting intervals found in the literature. It is important to emphasize that the purpose of these examples is to illustrate the mathematical approach to find expressions for the upper bound $P_{\max}$ of the tuning probability. As mentioned in Section 1, the actual lengths of the life-permitting intervals might change in the future, and thereby affect the values of $P_{\max}$.

4.1 The Gravitational Constant in $\mathbb{R}^+$.

The quantity of interest in terms of fine-tuning is the gravitational constant $G_{\text{obs}} = 6.67408 \times 10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2}$. Thus G_{obs} is an observation of the random variable G. However, in this example G_{obs} itself is not considered, but rather a ratio $x = G_{\text{obs}}/d$ between G_{obs} and some other constant of nature $d = D_{\text{obs}}$. This ratio is then an observation of the random quantity $X = G/D$, and as will seen below, for one particular choice of D the ratio X is essentially equivalent to the critical density of the universe at the Planck time.

According to the first step, in order to determine the appropriate sample space, gravity is assumed to be an attraction force, i.e. $G_{\text{obs}} > 0$, and the other constant of nature d is non-negative. Therefore x must be a non-negative real number as well. Thus the sample space to be considered is $\Omega = \mathbb{R}^+$.

For the second step —determining the right distribution— recall that the untruncated distribution F of X is of interest. Since X takes on values in $\mathbb{R}^+$,

$$P(X \leq x) = F(x;\mu) = F\left(\frac{x}{\mu};1\right) \tag{4.1}$$

for some distribution F (chosen below) with scale parameter $\theta = \mu$.

For the third step, several possible life-permitting intervals of the ratio $x = G_{\text{obs}}/d$ between G_{obs} and some other constant of nature d are now illustrated. In all of these cases the life-permitting interval has the form defined in (3.4); i.e.,

$$\ell_X = [x - \delta, x + \delta] = x \cdot [1 - \varepsilon, 1 + \varepsilon], \tag{4.2}$$

where δ is a positive number usually small, and $\varepsilon = \delta/x$, half of the relative size, is a dimensionless small number. As we will see below, our tuning results will confirm (3.5) and be expressed in terms of this dimensionless ε.

When $D = H^2$ is the random variable corresponding to the Hubble's constant squared at the Planck time, with observed value $d = H_{\text{Pl}}^2$, the first Friedmann equation (assuming Λ_{bare}, the cosmological constant, to be 0) is $x = G_{\text{obs}}/H_{\text{Pl}}^2 = 3/(8\pi\rho_{\text{crit}})$, where ρ_{crit} is the critical density of the universe at the Planck time. It then follows from Davies [2, pp. 88-89] that $\varepsilon = 10^{-60}$. We consider H^2 at the Planck time because we are looking at initial conditions of the universe, and the Planck time is as early as our theories go. Accordingly, we are varying our usual "obs" notation to avoid confusion with the value of the Hubble constant today.

Let $d = \Lambda_{\text{vac}}$, the dynamical contribution from vacuum energy to the cosmological constant. Under the Weinberg-Salam electroweak theory, $G_{\text{obs}}/\Lambda_{\text{vac}} = -\sqrt{2}c^4 g_w/\pi m_\phi^2$, where c is the speed of light, g_w is the weak force constant and m_ϕ is the mass of the scalar particles. In this scenario, Davies [2, p. 107] suggests $\varepsilon = 10^{-50}$ or even 10^{-100} in the case of grand unified theories. In fact, taking $\Lambda_{\text{bare}} \neq 0$, we arrive at the cosmological constant problem in which the observed total cosmological constant, $\Lambda_{\text{tot}} = \Lambda_{\text{bare}} + \Lambda_{\text{vac}}$ is 10^{120} times smaller than the predicted value of Λ_{vac} [8, Ch. 5].

Other ratios are possible. For instance, the ratio of constant of gravity to the constant of electromagnetic force is common in the fine-tuning literature [10, Ch. 3], [8, Ch. 4], [38, Sect. IV]. Whatever the ratio, or the constant, the general three-step method can be applied.

Once the life-permitting interval ℓ_X in (4.2) between G_{obs} and the other constant of nature d has been determined, (3.3) can be used to obtain the tuning probability.

4.1.1 The maximum entropy principle applied to a ratio G/D in $\mathbb{R}^+$.

Let us first assume that the distribution of X itself is chosen according to the MaxEnt principle. Since the sample space is $\Omega = \mathbb{R}^+$, it follows from Table 1 that $F \sim \text{Exp}(1/\mu)$. This gives a probability of tuning,

$$\begin{aligned} P(X \in \ell_X) &= \exp(-(x-\delta)/\mu) - \exp(-(x+\delta)/\mu) \\ &= \exp(-(x-\delta)/\mu)[1 - \exp(-2\delta/\mu)] \\ &= 2e^{-x/\mu}\sinh(\delta/\mu) \\ &= P(\mu, \varepsilon), \end{aligned} \tag{4.3}$$

where $\varepsilon = \delta/x$. Although $P(\mu, \varepsilon)$ depends on μ, it is uniformly small in μ (while keeping ε fixed). That is, according to (3.3), and in agreement with (3.5), the degree of tuning,

$$P_{\max}(\varepsilon) = \sup_{\mu > 0} P(\mu, \varepsilon), \tag{4.4}$$

is of the same order as ε. $P_{\max}(\varepsilon)$ in (4.4) can be calculated analytically by first solving for $dP(\mu, \varepsilon)/d\mu = 0$, and then inserting this value of μ into (4.3). In fact, $P(\cdot; \varepsilon)$ has a maximum at

$$\mu = \frac{2x\varepsilon}{\log((1+\varepsilon)/(1-\varepsilon))}.$$

A Taylor expansion of the denominator of the maximizing μ gives

$$\mu \approx x(1-\varepsilon) \approx x, \tag{4.5}$$

so that

$$\begin{aligned} P_{\max}(\varepsilon) &\approx P(x; \varepsilon) \\ &= 2e^{-1}\sinh(\varepsilon) \\ &\approx 0.7358 \cdot \varepsilon, \end{aligned} \tag{4.6}$$

where ε equals 10^{-50}, 10^{-60}, 10^{-100} or other values depending, respectively, on whether d is the Hubble constant squared, the energy of the quantum vacuum, or any other relevant ratio under consideration. Since $\varepsilon \ll 1$, there is extreme fine tuning in these cases.

4.1.2 The maximum entropy principle applied to each of two random variables G and D in $\mathbb{R}^+$ that form a ratio.

As a second application of (3), assume that $X = G/D$, where $G > 0$ and $D > 0$ are independent random variables, with distributions chosen according to the MaxEnt principle, i.e., $G \sim \text{Exp}(1/\mu_G)$ and $D \sim \text{Exp}(1/\mu_D)$, where $\mu_G = E(G)$ and $\mu_D = E(D)$ are two parameters that vary independently. Then X has a ratio distribution with density

$$\begin{aligned} f_X(y;\mu) &= \int_0^\infty u f_D(u;\mu_D) f_G(uy;\mu_G) du \\ &= \frac{1}{\mu} \cdot \frac{1}{(1+y/\mu)^2} \end{aligned}$$

for $y > 0$ and $\mu = \mu_G/\mu_D$. This gives a tuning probability

$$\begin{aligned} P(X \in \ell_X) &= \frac{1}{\mu} \int_{x-\delta}^{x+\delta} \frac{1}{(1+y/\mu)^2} dy \\ &= 2\delta \cdot \frac{\mu}{(\mu + x - \delta)(\mu + x + \delta)} \\ &= P(\varepsilon;\mu), \end{aligned} \tag{4.7}$$

where $\varepsilon = \delta/x$. Since the tuning probability in (4.7) only depends on μ_G and μ_D through their ratio $\mu = \mu_G/\mu_D$, according to (3.3), $P_{\max}$ is obtained by maximizing (4.7) with respect to $\mu > 0$. This maximum is

$$\mu = \sqrt{(x-\delta)(x+\delta)} = x\sqrt{1-\varepsilon^2} \approx x. \tag{4.8}$$

Inserting (4.8) into (4.7) we thus obtain the upper bound of the tuning probability,

$$\begin{aligned} P_{\max}(\varepsilon) &= P(x\sqrt{1-\varepsilon^2};\varepsilon) \\ &\approx 2x\varepsilon \cdot \frac{x}{4x^2} \\ &= \frac{\varepsilon}{2}. \end{aligned} \tag{4.9}$$

Note that (4.9) is minuscule being slightly smaller than the corresponding upper bound in (4.6), where the MaxEnt principle was applied directly to the distribution of the ratio X. Again, extreme fine tuning is seen.

4.2 The gravitational constant in $\mathbb{R}$.

In the previous example the only possibility considered was for gravity to be a nonnegative number. Were G allowed to be negative, both repulsion and attraction would be possible. Were G allowed to be 0, gravity would be a neutral force. In this scenario, according to the first step, G_{obs} must be thought of as a constant whose possible values constitute the whole real line $\mathbb{R}$. The same is true for the ratio $x = G_{\text{obs}}/d$ whether d is negative or positive.

As in the previous example, the MaxEnt principle will now be applied to the distribution of X, or separately to the distributions of G and D.

4.2.1 The maximum entropy principle applied to a ratio G/D in $\mathbb{R}$.

Assume first that the MaxEnt principle is applied to X. Table 1 says that in this case the distribution that better explains X is a normal $\mathcal{N}(\mu, \sigma^2)$. As in the previous example, x can be regarded as the midpoint of ℓ_X. This leads to a probability of the event $\{X \in \ell_X\}$ (step III):

$$P(X \in \ell_X) = \Phi[(x + \delta - \mu)/\sigma)] - \Phi[(x - \delta - \mu)/\sigma] = P(\mu, \sigma; \varepsilon), \tag{4.10}$$

where $\Phi(\cdot)$ is the standard normal distribution and $\varepsilon = \delta/x$. Generally, (4.10) is small for large σ regardless of the value of μ. However, by choosing $\mu = x$ in (4.10) and then letting $\sigma \to 0$, (3.3) attains the largest possible value

$$P_{\max} = \sup_{\mu\in\mathbb{R},\sigma>0} P(\mu, \sigma; \varepsilon) = 1. \tag{4.11}$$

In this case, since μ/σ is unrestricted, the constant of proportionality C in (3.5) is unbounded. This makes the tuning coarse.

In order to have a smaller value of $P_{\max}$, additional restrictions to the class of prior distributions are necessary. For instance, under the assumption that neither positive nor negative values of X should be favored a priori, enforcing $\mu = 0$ is natural. Then (4.10) takes the form

$$P(\sigma; \varepsilon) \approx \phi(x/\sigma)\frac{2\delta}{\sigma}, \tag{4.12}$$

where $\phi(\cdot) = \Phi'(\cdot)$ is the density function of a standard normal. This expression will have a very small upper bound, uniformly in σ, since

$$\begin{aligned} P_{\max}(\varepsilon) &= \sup_{\sigma>0} P(\sigma; \varepsilon) \\ &\approx \frac{2\delta}{x} \max_{\sigma>0} \left\{\phi(x/\sigma)\frac{x}{\sigma}\right\} \\ &= \frac{2\delta\phi(1)}{x} \\ &\approx \frac{2\delta \cdot 0.242}{x} \\ &= 0.484 \cdot \varepsilon, \end{aligned} \tag{4.13}$$

where the second equality is obtained by maximizing $\phi(x/\sigma)(x/\sigma)$ with respect to $\sigma > 0$. This is equivalent to maximizing $\phi(z)z$ with respect to $z > 0$, which achieves a maximum at $\phi(1) = e^{-1/2}/\sqrt{2\pi} \approx 0.242$. See Figure 1 below, for an illustration of the bounds in (4.11) and (4.13). Again, we see extreme fine-tuning provided $\mu \notin \ell_G$.

4.2.2 The maximum entropy principle applied to each of two random variables G and D in $\mathbb{R}$ that form a ratio.

We now turn to the second approach of choosing the distribution F of $X = G/D$, by using the MaxEnt principle to G and D separately, where G and D are both allowed to take on negative and positive values. In order not to favor either positive or negative values of G or D, assume that $G \sim \mathcal{N}(0, \sigma_G^2)$ and $D \sim \mathcal{N}(0, \sigma_D^2)$. Then X has a Cauchy distribution with density

$$\begin{aligned} f_X(y; \sigma) &= \int_{-\infty}^{\infty} |z| f_D(z; \sigma_D) f_G(yz; \sigma_G) dz \\ &= \frac{1}{\sigma\pi} \frac{1}{(1 + (y/\sigma)^2)}, \end{aligned}$$

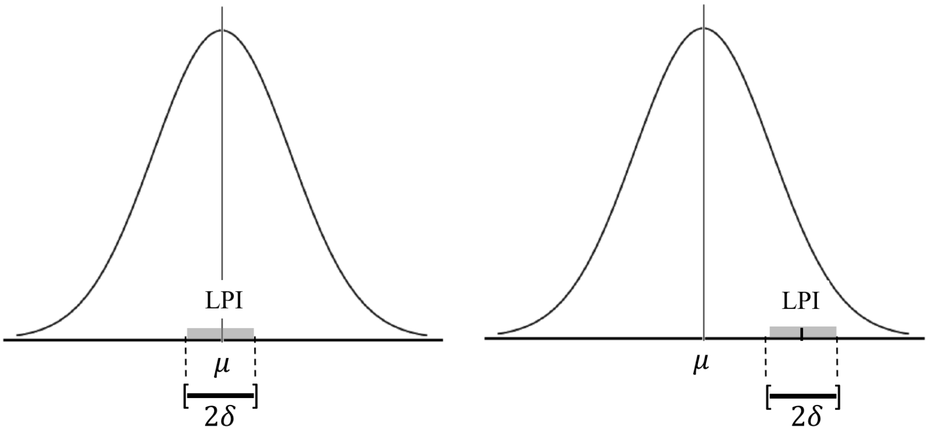

Figure 1. When the variance σ^2 of the prior distribution of X approaches 0, the normal distribution approaches a Dirac delta measure at μ; thus $\mu \in \ell_X$, implies $P_{\max} = 1$ (left). On the other hand, when $\mu \notin \ell_X$, where ℓ_X is the life-permitting interval (LPI), $P[\ell_G]$ will go to zero either when $\sigma \to 0$ or when $\sigma \to \infty$. Therefore $P_{\max}$ is strictly less than 1 (right). For the figure at the left, the tuning is coarse, whereas for the figure at the right it is fine.

for $y \in \mathbb{R}$, with $\sigma = \sigma_G/\sigma_D$. This gives a tuning probability

$$\begin{aligned} P(\sigma;\varepsilon) &= \int_{x-\delta}^{x+\delta} f_X(z;\sigma)dz \\ &\approx 2\delta f_X(x;\sigma) \\ &= \frac{2x\varepsilon}{\sigma}\frac{1}{\pi(1+(x/\sigma)^2)} \end{aligned} \tag{4.14}$$

that only depends on σ_G and σ_D through their ratio σ. Maximizing (4.14) with respect to $\sigma > 0$ is equivalent to taking the maximum with respect to $z = x/\sigma > 0$. Consequently

$$\begin{aligned} P_{\max}(\varepsilon) &\approx \frac{2\varepsilon}{\pi}\max_{z>0}\frac{z}{1+z^2} \\ &= \frac{\varepsilon}{\pi}. \end{aligned}$$

Interestingly, the Cauchy distribution itself (with scale parameter σ) is MaxEnt over $\mathbb{R}$ under the restriction that $E(\ln(1 + X^2/\sigma^2)) = 2\ln 2$ [35].

4.2.3 The maximum entropy principle applied to each of two constants G in $\mathbb{R}$ and D in $\mathbb{R}^+$, that form a ratio.

Another possibility is when $G \in \mathbb{R}$ and $D > 0$ are chosen to have MaxEnt distributions $G \sim \mathcal{N}(\mu_G, \sigma_G^2)$ and $D \sim \mathrm{Exp}(1/\mu_D)$. As before, we assume $\mu_G = 0$. Then $X = G/D$ will have a symmetric ratio distribution with scale parameter $\sigma = \sigma_G/\mu_D$. The tuning probability is obtained similarly as in (4.14), although the distribution of X is no longer Cauchy.

4.3 Amplitude of primordial fluctuations

The amplitude of primordial fluctuations, $q = Q_{\text{obs}} \approx 2 \times 10^{-5}$, is here the dimensionless value of interest [24]. Notice that in contrast to Examples 4.1 and 4.2, q is here a constant of nature rather than a ratio between two constants of nature. Rees [10, p. 128] writes:

> "If $[Q_{\rm obs}]$ were smaller than 10^{-6}, gas would never condense into gravitationally bound structures at all, and such a universe would remain forever dark and featureless, even if its initial 'mix' of atoms, dark energy and radiation were the same as our own.
>
> On the other hand, a universe where $[Q_{\rm obs}]$ were substantially larger than 10^{-5} – where the initial 'ripples' were replaced by large-amplitude waves – would be a turbulent and violent place. Regions far bigger than galaxies would condense early in its history. They wouldn't fragment into stars but would instead collapse into vast black holes, each much heavier than an entire cluster of galaxies in our universe... Stars would be packed too close together and buffeted too frequently to retain stable planetary systems."

Adams [7] gives an even larger life-permitting interval than Rees: $10^{-6} \le Q_{\rm obs} \le 10^{-2}$, but as we will see, there is little difference in our overall conclusion that the amplitude of primordial fluctuations is coarsely tuned. Following the three-step procedure, the first step is to determine the right sample space. As in the first example, the corresponding domain is the nonnegative reals $\mathbb{R}^+$, since fluctuations cannot be negative. Thus, from Table 1, we see that $Q \sim \text{Exp}(1/\mu)$, provided the mean μ is given. The third step is to calculate the probability of the life-permitting interval $\ell_Q = (10^{-6}, 10^{-5})$

$$P(Q \in \ell_Q) = \exp\left(-10^{-6}/\mu\right) - \exp\left(-10^{-5}/\mu\right) = P(\mu), \tag{4.15}$$

that depends on μ. In this case, according to (3.3), the maximal value is $P_{\max} = \max_{\mu>0} P(\mu)$. This maximization is similar to the one given in Example 4.1.1. Putting $a = 10^{-6}$ and $b = 10^{-5}$, the maximum tuning probability is obtained for

$$\mu = \frac{b-a}{\log(b/a)} = \frac{9 \times 10^{-6}}{\log(10)}. \tag{4.16}$$

Inserting (4.16) into (4.15) we obtain a rather large probability.

$$\begin{aligned} P_{\max} &= 10^{-1/9} - 10^{-10/9} \\ &\approx 0.697. \end{aligned} \tag{4.17}$$

This is surprising given that there are claims of a much higher degree of tuning [39]. This reveals that there looks to be a coarse tuning on the amplitude of the primordial fluctuations according to the MaxEnt model. From a mathematical point of view, the coarse tuning of Q is not too surprising though, since the upper limit of ℓ_Q is one order of magnitude larger than its lower limit.

4.4 Other constants or ratios

In Sections 4.1—4.3 we applied the MaxEnt principle either to ratios involving the gravitational constant (Examples 1-2) or to the amplitude of primordial fluctuations (Example 3). A number of other examples can be given. For instance, from Example 1, the ratio $x = \Lambda_{\rm vac}/\Lambda_{\rm bare}$ of the two terms involved in the cosmological constant is contained within an interval ℓ_X that is centered around -1, with a relative half length of $\varepsilon = 10^{-120}$.

5 Discussion

In this paper a Bayesian statistical procedure has been devised for calculating an upper bound $P_{\max}$ for the probability that a constant of nature, or the ratio of two constants of nature, belongs to a life permitting interval. This upper bound is invariant with respect to change of physical units, and under certain assumptions it is small whenever the life-permitting interval is small on a relative scale, corresponding to a small value of ε. We obtain $P_{\max}$ through a three-steps procedure, where I) the sample space is determined, II) the finite-dimensional class of distributions is found for this sample space by applying the maximum entropy principle, and III) the tuning probability is maximized over this finite-dimensional class of distributions.

In particular, for constants of nature or ratios that take on positive values, a one-dimensional class of exponential MaxEnt distributions is used. The upper bound of the tuning probability is then of the same order as ε, the length of half the life-permitting interval divided by its mid point. The proposed procedure guarantees a tuning probability that will be fine whenever the life-permitting interval is small in comparison to the size of its mid point, i.e. when ε is small.

For real-valued constants of nature, or ratios between such constants of nature, the class of normal MaxEnt distributions has two parameters: a location and scale parameter. Without further restrictions, the upper bound of the tuning probability is 1. However, the upper bound of the tuning probability will be small when:

a) The location parameter μ of a real-valued quantity's prior distribution is put to 0, so that a symmetric normal distribution with standard deviation σ is assumed.

b) The life-permitting interval does not contain 0, and ε, the half length of the interval divided by the modulus of the mid point, is small.

More generally, in order for tuning to be fine a) can be relaxed. Consider for instance the scenario of Section 4.2.1, where a normal MaxEnt prior distribution $\mathcal{N}(\mu, \sigma^2)$ was applied to a real-valued constant X, or to the ratio X of two such constants. In that case it is possible to replace a) by the weaker requirement that the signal-to-noise ratio $\mathrm{SNR} = \mu^2/\sigma^2$ of the prior distribution satisfies

$$\mathrm{SNR} \leq \mathrm{SNR}_{\max} \ll \varepsilon^{-2}. \tag{5.1}$$

Indeed, suppose equation (21) holds with $\mathrm{SNR}_{\max} > 0$ and that condition b) is satisfied. Then it can be shown that the degree of tuning (3) for a real-valued constant of nature, or a real-valued ratio of two such constants, is given by

$$P_{\max} = \max\left\{P(\mu, \sigma; \varepsilon);\ \frac{\mu^2}{\sigma^2} \leq \mathrm{SNR}_{\max}\right\} \approx 2\varepsilon\sqrt{\mathrm{SNR}_{\max}}\phi(0) \ll 1, \tag{5.2}$$

where $\delta = x\varepsilon$, x is the observed value of X, and $P(\mu, \sigma; \varepsilon)$ is the tuning probability defined in (4.10). Thus, for the approach used in Section 4.2.1, it follows from (5.2) that the tuning will be fine under very general conditions also for real-valued constants of nature, or for real-valued ratios of such constants. In the context of equation (3.5), the constant of proportionality of the tuning probability, $C = 2\sqrt{\mathrm{SNR}_{\max}}\phi(0)$, depends on how large the class of prior distributions of X is, in terms of the square root of the maximal signal-to-noise ratio.

A natural extension of this work is to find upper bounds for the joint tuning probability of several constants of nature. This would in turn provide a lower bound for the expected number of parallel universes needed in order to obtain one that permits life.

References

[1] B. Carter. Large Number Coincidences and the Anthropic Principle in Cosmology. In M. S. Longhair, editor, *Confrontation of Cosmological Theories with Observational Data*, pages 291–298. D. Reidel, 1974. URL https://www.doi.org/10.1017/S0074180900235638.

[2] P. Davies. *The Accidental Universe*. Cambridge University Press, 1982.

[3] S. Hawking. *A Brief History of Time*. Bantam Books, 1988.

[4] F. Hoyle. The Universe: Past and Present Reflections. *Annual Review of Astronomy and Astrophysics*, 20(1):1–36, 1982. URL https://doi.org/10.1146/annurev.aa.20.090182.000245.

[5] S. Weinberg. Life in the Universe. *Scientific American*, 271(4):44–49, 1994. URL http://www.jstor.org/stable/24942868.

[6] J. D. Barrow and F. J. Tipler. *The Anthropic Cosmological Principle*. Oxford University Press, 1988.

[7] F. C. Adams. The degree of fine-tuning in our universe —and others. *Physics Reports*, 807(15): 1–111, May 2019. URL http://www.doi.org/10.1016/j.physrep.2019.02.001.

[8] G. F. Lewis and L. A. Barnes. *A Fortunate Universe: Life In a Finely Tuned Cosmos*. Cambridge University Press, 2016.

[9] M. Tegmark, A. Aguirre, M. Rees, and F. Wilczek. Dimensionless constants, cosmology, and other dark matters. *Phys. Rev. D*, 73(2):023505, 2006. URL https://doi.org/10.1103/PhysRevD.73.023505.

[10] M. J. Rees. *Just Six Numbers: The Deep Forces That Shape The Universe*. Basic Books, 2000.

[11] L. A. Barnes. A Reasonable Little Question: A Formulation of the Fine-Tuning Argument. *Ergo*, 6(42):1220–1257, 2019-2020. URL https://doi.org/10.3998/ergo.12405314.0006.042.

[12] M. Sandora. Multiverse Predictions for Habitability: The Number of Stars and Their Properties. *Universe*, 5(6):149, 2019. URL https://www.doi.org/10.3390/universe5060149.

[13] M. Sandora. Multiverse Predictions for Habitability: Number of Potentially Habitable Planets. *Universe*, 5(6):157, 2019. URL https://www.doi.org/10.3390/universe5060157.

[14] M. Sandora. Multiverse Predictions for Habitability: Fraction of Planets That Develop Life. *Universe*, 5(7):171, 2019. URL https://www.doi.org/10.3390/universe5070171.

[15] M. Sandora. Multiverse Predictions for Habitability: Fraction of Life That Develops Intelligence. *Universe*, 5(7):175, 2019. URL https://www.doi.org/10.3390/universe5070175.

[16] R. Collins. The Teleological Argument: An Exploration of the Fine-Tuning of the Universe. In W. L. Craig and J. P. Moreland, editors, *Blackwell Companion to Natural Theology*, pages 202–281. Wiley-Blackwell, 2012. URL https://doi.org/10.1002/9781444308334.ch4.

[17] T. McGrew, L. McGrew, and E. Vestrup. Probabilities and the Fine-Tuning Argument: A Sceptical View. *Mind, New Series*, 110(440):1027–1037, October 2001. URL https://doi.org/10.1093/mind/110.440.1027.

[18] J. Bernoulli. *Ars Conjectandi*. Thurneysen Brothers, 1713.

[19] W. A. Dembski and R. J. Marks II. Bernoulli's Principle of Insufficient Reason and Conservation of Information in Computer Search. In *Proc. of the 2009 IEEE International Conference on Systems, Man, and Cybernetics. San Antonio, TX*, pages 2647–2652, October 2009. URL https://doi.org/10.1109/ICSMC.2009.5346119.

[20] W. Tschirk. The principle of indifference does not lead to contradictions. *International Journal of Statistics and Probability*, 5(4):79, 2016. URL https://www.doi.org/10.5539/ijsp.v5n4p79.

[21] R. Penrose. *The Road to Reality*. Random House, 2004.

[22] R. Penrose. Time-Assymetry and Quantum Gravity. In C. J. Isham, R. Penrose, and D. W. Sciama, editors, *Quantum Gravity 2: A Second Oxford Symposium*, pages 244–272. Oxford Univeristy Press, 1981.

[23] H. Martel, P. R. Shapiro, and S. Weinberg. Likely values of the cosmological constant. *The Astrophysical Journal*, 492(1):29–40, 1998. URL https://doi.org/10.1086%2F305016.

[24] L. A. Barnes. The Fine Tuning of the Universe for Intelligent Life. *Publications of the Astronomical Society of Australia*, 29(4):529–564, 2011. URL https://doi.org/10.1071/AS12015.

[25] L. A. Barnes. Testing the Multiverse: Bayes, Fine-Tuning and Typicality. In K. Chamcham, J. Silk, J. D. Barrow, and S. Saunders, editors, *The Philosophy of Cosmology*, pages 447–466. Cambridge University Press, 2017. URL https://doi.org/10.1017/9781316535783.023.

[26] L. A. Barnes. Fine-tuning in the context of Bayesian theory testing. *European Journal for Philosophy of Science*, 8(2):253–269, 2018. URL https://doi.org/10.1007/s13194-017-0184-2.

[27] E. T. Jaynes. *Probability Theory: The Logic of Science*. Cambridge University Press, 2003.

[28] J. Berger. *Statistical Decision Theory and Bayesian Analysis*. Springer, 1985.

[29] E. T. Jaynes. Information theory and statistical mechanics. *Physical Review*, 106(4):620–630, May 1957. URL https://doi.org/10.1103/PhysRev.106.620.

[30] E. T. Jaynes. Information theory and statistical mechanics ii. *Physical Review*, 108(2):171–190, 1957. URL https://doi.org/10.1103/PhysRev.108.171.

[31] D. A. Díaz-Pachón and R. J. Marks II. Generalized active information: Extensions to unbounded domains. *BIO-Complexity*, 2020(3):1–6, 2020. URL https://doi.org/10.5048/BIO-C.2020.3.

[32] L. McGrew and T. McGrew. On the Rational Reconstruction of the Fine-Tuning Argument. *Philosophia Christi*, 7(2):423–441, 2005. URL https://doi.org/10.5840/pc20057235.

[33] K. Conrad. Probability Distributions and Maximal Entropy, 2005. URL http://www.math.uconn.edu/~kconrad/blurbs/analysis/entropypost.pdf.

[34] T. M. Cover and J. A. Thomas. *Elements of Information Theory*. Wiley, second edition, 2006.

[35] S. Y. Park and A. K. Bera. Maximum entropy autoregressive conditional heteroskedasticity model. *J. of Econometrics*, 150:219–230, 2009. URL https://www.doi.org/10.1016/j.jeconom.2008.12.014.

[36] G. Casella and R. L. Berger. *Statistical Inference*. Cengage Learning, second edition, 2006.

[37] S. Thorvaldsen and O. Hössjer. Using statistical methods to model the fine-tuning of molecular machines and systems. *Journal of Theoretical Biology*, 501:110352, 2020. ISSN 0022-5193. URL https://doi.org/10.1016/j.jtbi.2020.110352.

[38] J. Uzan. Varying Constants, Gravitation and Cosmology. *Living Reviews in Relativity*, 14(2), 2011. URL https:/www.doi.org/10.12942/lrr-2011-2.

[39] M. Tegmark and M. J. Rees. Why is the cosmic microwave background fluctuation level 10^{-5}. *The Astrophysical Journal*, 499(2):526–532, 1998. URL https://www.doi.org/10.1086/305673.